Section 6.2

71–72 Find (a) the intervals of increase or decrease, (b) the intervals of concavity, and (c) the points of inflection.

71. $f(x) = (1 - x)e^{-x}$

72. $f(x) = \dfrac{e^x}{x^2}$

73–75 Discuss the curve using the guidelines of Section 3.5.

73. $y = e^{-1/(x+1)}$

74. $y = e^{-x} \sin x, \quad 0 \leqslant x \leqslant 2\pi$

75. $y = 1/(1 + e^{-x})$

76. Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx}f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.
(a) Find $g'(0)$ and $g''(0)$ in terms of c.
(b) In terms of k, find an equation of the tangent line to the graph of h at the point where $x = 0$.

77. A *drug response curve* describes the level of medication in the bloodstream after a drug is administered. A surge function $S(t) = At^pe^{-kt}$ is often used to model the response curve, reflecting an initial surge in the drug level and then a more gradual decline. If, for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate the times corresponding to the inflection points and explain their significance. If you have a graphing device, use it to graph the drug response curve.

78. After an antibiotic tablet is taken, the concentration of the antibiotic in the bloodstream is modeled by the function

$$C(t) = 8(e^{-0.4t} - e^{-0.6t})$$

where the time t is measured in hours and C is measured in μg/mL. What is the maximum concentration of the antibiotic during the first 12 hours?

79. After the consumption of an alcoholic beverage, the concentration of alcohol in the bloodstream (blood alcohol concentration, or BAC) surges as the alcohol is absorbed, followed by a gradual decline as the alcohol is metabolized. The function

$$C(t) = 1.35te^{-2.802t}$$

models the average BAC, measured in mg/mL, of a group of eight male subjects t hours after rapid consumption of 15 mL of ethanol (corresponding to one alcoholic drink). What is the maximum average BAC during the first 3 hours? When does it occur?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

80–81 Draw a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

80. $f(x) = e^{\cos x}$

81. $f(x) = e^{x^3 - x}$

82. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

83–94 Evaluate the integral.

83. $\int_0^1 (x^e + e^x)\, dx$

84. $\int_{-5}^5 e\, dx$

85. $\int_0^2 \dfrac{dx}{e^{\pi x}}$

86. $\int x^2 e^{x^3}\, dx$

87. $\int e^x \sqrt{1 + e^x}\, dx$

88. $\int \dfrac{(1 + e^x)^2}{e^x}\, dx$

89. $\int (e^x + e^{-x})^2\, dx$

90. $\int e^x(4 + e^x)^5\, dx$

91. $\int \dfrac{e^u}{(1 - e^u)^2}\, du$

92. $\int e^{\sin\theta} \cos\theta\, d\theta$

93. $\int_1^2 \dfrac{e^{1/x}}{x^2}\, dx$

94. $\int_0^1 \dfrac{\sqrt{1 + e^{-x}}}{e^x}\, dx$

95. Find, correct to three decimal places, the area of the region bounded by the curves $y = e^x$, $y = e^{3x}$, and $x = 1$.

96. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, and $f'(0) = 2$.

97. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

98. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$.

99. The **error function**

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

for any positive power p. So for large x, the values of $\ln x$ are very small compared with x^p. (See Exercise 72.)

x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
$\sqrt{x}$	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\dfrac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

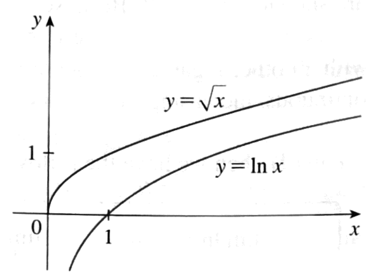

FIGURE 5

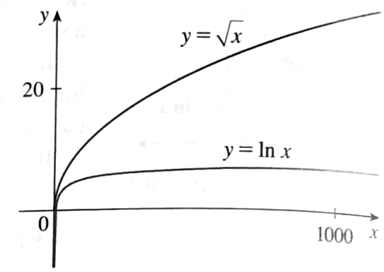

FIGURE 6

6.3 EXERCISES

1. (a) How is the logarithmic function $y = \log_b x$ defined?
   (b) What is the domain of this function?
   (c) What is the range of this function?
   (d) Sketch the general shape of the graph of the function $y = \log_b x$ if $b > 1$.

2. (a) What is the natural logarithm?
   (b) What is the common logarithm?
   (c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

3–8 Find the exact value of each expression.

3. (a) $\log_2 32$ (b) $\log_8 2$

4. (a) $\log_5 \frac{1}{125}$ (b) $\ln(1/e^2)$

5. (a) $e^{\ln 4.5}$ (b) $\log_{10} 0.0001$

6. (a) $\log_{1.5} 2.25$ (b) $\log_5 4 - \log_5 500$

7. (a) $\log_{10} 40 + \log_{10} 2.5$
   (b) $\log_8 60 - \log_8 3 - \log_8 5$

8. (a) $e^{-\ln 2}$ (b) $e^{\ln(\ln e^3)}$

9–12 Use the properties of logarithms to expand the quantity.

9. $\ln \sqrt{ab}$

10. $\log_{10} \sqrt{\dfrac{x-1}{x+1}}$

11. $\ln \dfrac{x^2}{y^3 z^4}$

12. $\ln\left(s^4 \sqrt{t\sqrt{u}}\right)$

13–18 Express the quantity as a single logarithm.

13. $2 \ln x + 3 \ln y - \ln z$

14. $\log_{10} 4 + \log_{10} a - \frac{1}{3} \log_{10}(a + 1)$

15. $\ln 10 + 2 \ln 5$

16. $\ln 3 + \frac{1}{3} \ln 8$

17. $\frac{1}{3} \ln(x + 2)^3 + \frac{1}{2}[\ln x - \ln(x^2 + 3x + 2)^2]$

18. $\ln b + 2 \ln c - 3 \ln d$

19. Use Formula 7 to evaluate each logarithm correct to six decimal places.
    (a) $\log_5 10$ (b) $\log_3 57$ (c) $\log_2 \pi$

20–22 Use Formula 7 to graph the given functions on a common screen. How are these graphs related?

20. $y = \log_2 x$, $y = \log_4 x$, $y = \log_6 x$, $y = \log_8 x$

21. $y = \log_{1.5} x$, $y = \ln x$, $y = \log_{10} x$, $y = \log_{50} x$

22. $y = \ln x$, $y = \log_{10} x$, $y = e^x$, $y = 10^x$

23–24 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 2 and 3 and, if necessary, the transformations of Section 1.3.

23. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

24. (a) $y = \ln(-x)$ (b) $y = \ln|x|$

25–26
(a) What are the domain and range of f?
(b) What is the x-intercept of the graph of f?
(c) Sketch the graph of f.

25. $f(x) = \ln x + 2$ **26.** $f(x) = \ln(x - 1) - 1$

27–36 Solve each equation for x.

27. (a) $e^{7-4x} = 6$ (b) $\ln(3x - 10) = 2$

28. (a) $\ln(x^2 - 1) = 3$ (b) $e^{2x} - 3e^x + 2 = 0$

29. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

30. (a) $e^{3x+1} = k$ (b) $\log_2(mx) = c$

31. $e - e^{-2x} = 1$ **32.** $10(1 + e^{-x})^{-1} = 3$

33. $\ln(\ln x) = 1$ **34.** $e^{e^x} = 10$

35. $e^{2x} - e^x - 6 = 0$ **36.** $\ln(2x + 1) = 2 - \ln x$

37–38 Find the solution of the equation correct to four decimal places.

37. (a) $\ln(1 + x^3) - 4 = 0$ (b) $2e^{1/x} = 42$

38. (a) $2^{1-3x} = 99$ (b) $\ln\left(\dfrac{x + 1}{x}\right) = 2$

39–40 Solve each inequality for x.

39. (a) $\ln x < 0$ (b) $e^x > 5$

40. (a) $1 < e^{3x-1} < 2$ (b) $1 - 2\ln x < 3$

41. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

42. The velocity of a particle that moves in a straight line under the influence of viscous forces is $v(t) = ce^{-kt}$, where c and k are positive constants.
(a) Show that the acceleration is proportional to the velocity.
(b) Explain the significance of the number c.
(c) At what time is the velocity equal to half the initial velocity?

43. The geologist C. F. Richter defined the magnitude of an earthquake to be $\log_{10}(I/S)$, where I is the intensity of the quake (measured by the amplitude of a seismograph 100 km from the epicenter) and S is the intensity of a "standard" earthquake (where the amplitude is only 1 micron $= 10^{-4}$ cm). The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

44. A sound so faint that it can just be heard has intensity $I_0 = 10^{-12}$ watt/m^2 at a frequency of 1000 hertz (Hz). The loudness, in decibels (dB), of a sound with intensity I is then defined to be $L = 10\log_{10}(I/I_0)$. Amplified rock music is measured at 120 dB, whereas the noise from a motor-driven lawn mower is measured at 106 dB. Find the ratio of the intensity of the rock music to that of the mower.

45. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$.
(a) Find the inverse of this function and explain its meaning.
(b) When will the population reach 50,000?

46. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)
(a) Find the inverse of this function and explain its meaning.
(b) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

47–52 Find the limit.

47. $\lim_{x \to 3^+} \ln(x^2 - 9)$ **48.** $\lim_{x \to 2^-} \log_5(8x - x^4)$

49. $\lim_{x \to 0} \ln(\cos x)$ **50.** $\lim_{x \to 0^+} \ln(\sin x)$

51. $\lim_{x \to \infty} [\ln(1 + x^2) - \ln(1 + x)]$

52. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

53–54 Find the domain of the function.

53. $f(x) = \ln(4 - x^2)$ **54.** $g(x) = \log_2(x^2 + 3x)$

55–57 Find (a) the domain of f and (b) f^{-1} and its domain.

55. $f(x) = \sqrt{3 - e^{2x}}$ **56.** $f(x) = \ln(2 + \ln x)$

57. $f(x) = \ln(e^x - 3)$

58. (a) What are the values of $e^{\ln 300}$ and $\ln(e^{300})$?
(b) Use your calculator to evaluate $e^{\ln 300}$ and $\ln(e^{300})$. What do you notice? Can you explain why the calculator has trouble?

The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x \to 0} \ln(1+x)^{1/x}$$

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1+x)^{1/x} = 1$$

Then, by Theorem 1.8.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1+x)^{1/x}$$

8 $$e = \lim_{x \to 0} (1+x)^{1/x}$$

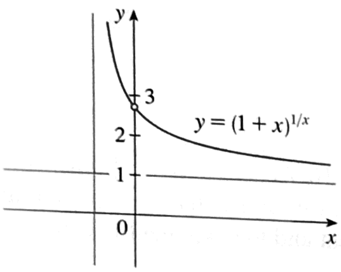

FIGURE 7

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

Formula 8 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 7 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

If we put $n = 1/x$ in Formula 8, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

9 $$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

6.4 EXERCISES

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_b x$.

2–26 Differentiate the function.

2. $f(x) = x \ln x - x$

3. $f(x) = \sin(\ln x)$

4. $f(x) = \ln(\sin^2 x)$

5. $f(x) = \ln \dfrac{1}{x}$

6. $y = \dfrac{1}{\ln x}$

7. $f(x) = \log_{10}(1 + \cos x)$

8. $f(x) = \log_{10} \sqrt{x}$

9. $g(x) = \ln(xe^{-2x})$

10. $g(t) = \sqrt{1 + \ln t}$

11. $F(t) = (\ln t)^2 \sin t$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $G(y) = \ln \dfrac{(2y+1)^5}{\sqrt{y^2 + 1}}$

14. $P(v) = \dfrac{\ln v}{1 - v}$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

16. $y = \ln|1 + t - t^3|$

17. $f(x) = x^5 + 5^x$

18. $g(x) = x \sin(2^x)$

19. $T(z) = 2^z \log_2 z$

20. $y = \ln(\csc x - \cot x)$

21. $y = \ln(e^{-x} + xe^{-x})$

22. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

23. $y = \tan[\ln(ax + b)]$

24. $y = \log_2(x \log_5 x)$

25. $G(x) = 4^{C/x}$

26. $F(t) = 3^{\cos 2t}$

6.4

(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.

(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).

(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

71–82 Evaluate the integral.

71. $\int_2^4 \frac{3}{x}\,dx$

72. $\int_0^3 \frac{dx}{5x+1}$

73. $\int_1^2 \frac{dt}{8-3t}$

74. $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

75. $\int_1^e \frac{x^2+x+1}{x}\,dx$

76. $\int \frac{\cos(\ln t)}{t}\,dt$

77. $\int \frac{(\ln x)^2}{x}\,dx$

78. $\int \frac{\cos x}{2+\sin x}\,dx$

79. $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

80. $\int \frac{e^x}{e^x+1}\,dx$

81. $\int_0^4 2^s\,ds$

82. $\int x 2^{x^2}\,dx$

---

83. Show that $\int \cot x\,dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 11.

84. Sketch the region enclosed by the curves

$$y = \frac{\ln x}{x} \quad \text{and} \quad y = \frac{(\ln x)^2}{x}$$

and find its area.

85. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{\sqrt{x+1}}$$

from 0 to 1 about the x-axis.

86. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2+1}$$

from 0 to 3 about the y-axis.

87. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 5.4.29.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

88. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

89. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

90. If $f(x) = e^x + \ln x$ and $h(x) = f^{-1}(x)$, find $h'(e)$.

91. For what values of m do the line $y = mx$ and the curve $y = x/(x^2+1)$ enclose a region? Find the area of the region.

92. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

93. Use the definition of derivative to prove that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

94. Show that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

6.2* The Natural Logarithmic Function

If your instructor has assigned Sections 6.2–6.4 (pp. 408–438), you need not read Sections 6.2*, 6.3*, and 6.4* (pp. 438–465).

In this section we define the natural logarithm as an integral and then show that it obeys the usual laws of logarithms. The Fundamental Theorem makes it easy to differentiate this function.

1 Definition The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$