$$\cdots \frac{\ }{22 \cdot 2^{22}} \qquad \frac{\ }{(7n+1)2^{7n+1}} + \cdots$$

This infinite series is the exact value of the definite integral, but since it is an alternating series, we can approximate the sum using the Alternating Series Estimation Theorem. If we stop adding after the term with $n = 3$, the error is smaller than the term with $n = 4$:

$$\frac{1}{29 \cdot 2^{29}} \approx 6.4 \times 10^{-11}$$

So we have

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx \approx \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374$$

11.9 EXERCISES

1. If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is 10, what is the radius of convergence of the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$? Why?

2. Suppose you know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about the following series? Why?

$$\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

3–10 Find a power series representation for the function and determine the interval of convergence.

3. $f(x) = \dfrac{1}{1+x}$

4. $f(x) = \dfrac{5}{1-4x^2}$

5. $f(x) = \dfrac{2}{3-x}$

6. $f(x) = \dfrac{4}{2x+3}$

7. $f(x) = \dfrac{x^2}{x^4+16}$

8. $f(x) = \dfrac{x}{2x^2+1}$

9. $f(x) = \dfrac{x-1}{x+2}$

10. $f(x) = \dfrac{x+a}{x^2+a^2}, \quad a > 0$

11–12 Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence.

11. $f(x) = \dfrac{2x-4}{x^2-4x+3}$

12. $f(x) = \dfrac{2x+3}{x^2+3x+2}$

13. (a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

What is the radius of convergence?

(b) Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}$$

(c) Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}$$

14. (a) Use Equation 1 to find a power series representation for $f(x) = \ln(1-x)$. What is the radius of convergence?
(b) Use part (a) to find a power series for $f(x) = x\ln(1-x)$.
(c) By putting $x = \frac{1}{2}$ in your result from part (a), express ln 2 as the sum of an infinite series.

15–20 Find a power series representation for the function and determine the radius of convergence.

15. $f(x) = \ln(5-x)$

16. $f(x) = x^2 \tan^{-1}(x^3)$

17. $f(x) = \dfrac{x}{(1+4x)^2}$

18. $f(x) = \left(\dfrac{x}{2-x}\right)^3$

19. $f(x) = \dfrac{1+x}{(1-x)^2}$

20. $f(x) = \dfrac{x^2+x}{(1-x)^3}$

11.9

21–24 Find a power series representation for f, and graph f and several partial sums $s_n(x)$ on the same screen. What happens as n increases?

21. $f(x) = \dfrac{x^2}{x^2 + 1}$

22. $f(x) = \ln(1 + x^4)$

23. $f(x) = \ln\left(\dfrac{1 + x}{1 - x}\right)$

24. $f(x) = \tan^{-1}(2x)$

25–28 Evaluate the indefinite integral as a power series. What is the radius of convergence?

25. $\displaystyle\int \frac{t}{1 - t^8}\,dt$

26. $\displaystyle\int \frac{t}{1 + t^3}\,dt$

27. $\displaystyle\int x^2 \ln(1 + x)\,dx$

28. $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx$

29–32 Use a power series to approximate the definite integral to six decimal places.

29. $\displaystyle\int_0^{0.3} \frac{x}{1 + x^3}\,dx$

30. $\displaystyle\int_0^{1/2} \arctan(x/2)\,dx$

31. $\displaystyle\int_0^{0.2} x\ln(1 + x^2)\,dx$

32. $\displaystyle\int_0^{0.3} \frac{x^2}{1 + x^4}\,dx$

33. Use the result of Example 7 to compute arctan 0.2 correct to five decimal places.

34. Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0$$

35. (a) Show that J_0 (the Bessel function of order 0 given in Example 4) satisfies the differential equation

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0$$

37. (a) Show that the funct

is a solution of the

(b) Show that $f(x) =$

38. Let $f_n(x) = (\sin nx)/n$ verges for all values o diverges when $x = 2r$ does the series $\Sigma f_n''(x$

39. Let

Find the intervals of

40. (a) Starting with the of the series

(b) Find the sum of

(i) $\sum_{n=1}^{\infty} nx^n$, $|x$

(c) Find the sum o

(i) $\sum_{n=2}^{\infty} n(n -$

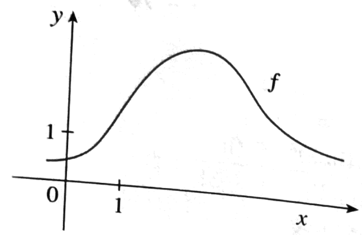
(ii) $\sum_{n=2}^{\infty} \frac{n^2 - n}{2^n}$

41. Use the power ser sion for π as the s

11.10 EXERCISES

1. If $f(x) = \sum_{n=0}^{\infty} b_n(x-5)^n$ for all x, write a formula for b_8.

2. The graph of f is shown.



(a) Explain why the series

$$1.6 - 0.8(x-1) + 0.4(x-1)^2 - 0.1(x-1)^3 + \cdots$$

is *not* the Taylor series of f centered at 1.

(b) Explain why the series

$$2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$$

is *not* the Taylor series of f centered at 2.

3. If $f^{(n)}(0) = (n+1)!$ for $n = 0, 1, 2, \ldots$, find the Maclaurin series for f and its radius of convergence.

4. Find the Taylor series for f centered at 4 if

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$$

What is the radius of convergence of the Taylor series?

5–10 Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a.

5. $f(x) = xe^x$, $a = 0$

6. $f(x) = \dfrac{1}{1+x}$, $a = 2$

7. $f(x) = \sqrt[3]{x}$, $a = 8$

8. $f(x) = \ln x$, $a = 1$

9. $f(x) = \sin x$, $a = \pi/6$

10. $f(x) = \cos^2 x$, $a = 0$

11–18 Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

11. $f(x) = (1-x)^{-2}$

12. $f(x) = \ln(1+x)$

13. $f(x) = \cos x$

14. $f(x) = e^{-2x}$

15. $f(x) = 2^x$

16. $f(x) = x\cos x$

17. $f(x) = \sinh x$

18. $f(x) = \cosh x$

19–26 Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

19. $f(x) = x^5 + 2x^3 + x$, $a = 2$

20. $f(x) = x^6 - x^4 + 2$, $a = -2$

21. $f(x) = \ln x$, $a = 2$

22. $f(x) = 1/x$, $a = -3$

23. $f(x) = e^{2x}$, $a = 3$

24. $f(x) = \cos x$, $a = \pi/2$

25. $f(x) = \sin x$, $a = \pi$

26. $f(x) = \sqrt{x}$, $a = 16$

27. Prove that the series obtained in Exercise 13 represents $\cos x$ for all x.

28. Prove that the series obtained in Exercise 25 represents $\sin x$ for all x.

29. Prove that the series obtained in Exercise 17 represents $\sinh x$ for all x.

30. Prove that the series obtained in Exercise 18 represents $\cosh x$ for all x.

31–34 Use the binomial series to expand the function as a power series. State the radius of convergence.

31. $\sqrt[4]{1-x}$

32. $\sqrt[3]{8+x}$

33. $\dfrac{1}{(2+x)^3}$

34. $(1-x)^{3/4}$

35–44 Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the given function.

35. $f(x) = \arctan(x^2)$

36. $f(x) = \sin(\pi x/4)$

37. $f(x) = x\cos 2x$

38. $f(x) = e^{3x} - e^{2x}$

39. $f(x) = x\cos(\frac{1}{2}x^2)$

40. $f(x) = x^2\ln(1+x^3)$

41. $f(x) = \dfrac{x}{\sqrt{4+x^2}}$

42. $f(x) = \dfrac{x^2}{\sqrt{2+x}}$

43. $f(x) = \sin^2 x$ $\left[\textit{Hint:} \text{ Use } \sin^2 x = \frac{1}{2}(1-\cos 2x).\right]$

44. $f(x) = \begin{cases} \dfrac{x - \sin x}{x^3} & \text{if } x \neq 0 \\ \dfrac{1}{6} & \text{if } x = 0 \end{cases}$

45–48 Find the Maclaurin series of f (by any method) and its radius of convergence. Graph f and its first few Taylor polynomials on the same screen. What do you notice about the relationship between these polynomials and f?

45. $f(x) = \cos(x^2)$

46. $f(x) = \ln(1+x^2)$

47. $f(x) = xe^{-x}$

48. $f(x) = \tan^{-1}(x^3)$

49. Use the Maclaurin series for $\cos x$ to compute $\cos 5°$ correct to five decimal places.