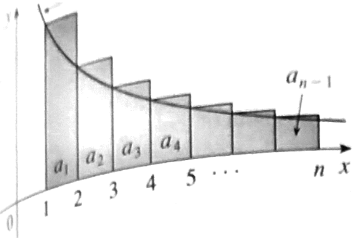

FIGURE 6

5 $$\int_1^n f(x)\,dx \leqslant a_1 + a_2 + \cdots + a_{n-1}$$

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then (4) gives

$$\sum_{i=2}^{n} a_i \leqslant \int_1^n f(x)\,dx \leqslant \int_1^\infty f(x)\,dx$$

since $f(x) \geqslant 0$. Therefore

$$s_n = a_1 + \sum_{i=2}^{n} a_i \leqslant a_1 + \int_1^\infty f(x)\,dx = M, \text{ say}$$

Since $s_n \leqslant M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geqslant s_n$$

since $a_{n+1} = f(n + 1) \geqslant 0$. Thus $\{s_n\}$ is an increasing bounded sequence and so it is convergent by the Monotonic Sequence Theorem (11.1.12). This means that $\sum a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\int_1^n f(x)\,dx \to \infty$ as $n \to \infty$ because $f(x) \geqslant 0$. But (5) gives

$$\int_1^n f(x)\,dx \leqslant \sum_{i=1}^{n-1} a_i = s_{n-1}$$

and so $s_{n-1} \to \infty$. This implies that $s_n \to \infty$ and so $\sum a_n$ diverges. ■

11.3 EXERCISES

1. Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^\infty \frac{1}{x^{1.3}}\,dx$$

What can you conclude about the series?

2. Suppose f is a continuous positive decreasing function for $x \geqslant 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x)\,dx \qquad \sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{6} a_i$$

3–8 Use the Integral Test to determine whether the series is convergent or divergent.

3. $\sum_{n=1}^{\infty} n^{-3}$

4. $\sum_{n=1}^{\infty} n^{-0.3}$

5. $\sum_{n=1}^{\infty} \frac{2}{5n - 1}$

6. $\sum_{n=1}^{\infty} \frac{1}{(3n - 1)^4}$

7. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

8. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

9–26 Determine whether the series is convergent or divergent.

9. $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

10. $\sum_{n=3}^{\infty} n^{-0.9999}$

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

12. $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \cdots$

13. $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \cdots$

14. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

15. $\sum_{n=1}^{\infty} \frac{\sqrt{n} + 4}{n^2}$

16. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1 + n^{3/2}}$

17. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$

18. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 2}$

19. $\sum_{n=1}^{\infty} \frac{n^3}{n^4 + 4}$

20. $\sum_{n=3}^{\infty} \frac{3n - 4}{n^2 - 2n}$

21. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

22. $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

23. $\sum_{k=1}^{\infty} ke^{-k}$

24. $\sum_{k=1}^{\infty} ke^{-k^2}$

25. $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^3}$

26. $\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$

27–28 Explain why the Integral Test can't be used to determine whether the series is convergent.

27. $\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$

28. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{1 + n^2}$

29–32 Find the values of p for which the series is convergent.

29. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

30. $\sum_{n=3}^{\infty} \frac{1}{n \ln n \, [\ln(\ln n)]^p}$

31. $\sum_{n=1}^{\infty} n(1 + n^2)^p$

32. $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

33. The Riemann zeta-function ζ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used in number theory to study the distribution of prime numbers. What is the domain of ζ?

34. Leonhard Euler was able to calculate the exact sum of the p-series with $p = 2$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(See page 760.) Use this fact to find the sum of each series.

(a) $\sum_{n=2}^{\infty} \frac{1}{n^2}$ (b) $\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$

(c) $\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

35. Euler also found the sum of the p-series with $p = 4$:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use Euler's result to find the sum of the series.

(a) $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$ (b) $\sum_{k=5}^{\infty} \frac{1}{(k-2)^4}$

36. (a) Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_{10} as an approximation to the sum of the series.
(b) Use (3) with $n = 10$ to give an improved estimate of the sum.
(c) Compare your estimate in part (b) with the exact value given in Exercise 35.
(d) Find a value of n so that s_n is within 0.00001 of the sum.

37. (a) Use the sum of the first 10 terms to estimate the sum of the series $\sum_{n=1}^{\infty} 1/n^2$. How good is this estimate?
(b) Improve this estimate using (3) with $n = 10$.
(c) Compare your estimate in part (b) with the exact value given in Exercise 34.
(d) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

38. Find the sum of the series $\sum_{n=1}^{\infty} ne^{-2n}$ correct to four decimal places.

39. Estimate $\sum_{n=1}^{\infty} (2n + 1)^{-6}$ correct to five decimal places.

40. How many terms of the series $\sum_{n=2}^{\infty} 1/[n(\ln n)^2]$ would you need to add to find its sum to within 0.01?

41. Show that if we want to approximate the sum of the series $\sum_{n=1}^{\infty} n^{-1.001}$ so that the error is less than 5 in the ninth decimal place, then we need to add more than $10^{11,301}$ terms!

CAS **42.** (a) Show that the series $\sum_{n=1}^{\infty} (\ln n)^2/n^2$ is convergent.
(b) Find an upper bound for the error in the approximation $s \approx s_n$.
(c) What is the smallest value of n such that this upper bound is less than 0.05?
(d) Find s_n for this value of n.

...calculator or a computer, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3+1} \approx 0.6864538$$

with error less than 0.00005.

11.4 EXERCISES

1. Suppose Σa_n and Σb_n are series with positive terms and Σb_n is known to be convergent.
   (a) If $a_n > b_n$ for all n, what can you say about Σa_n? Why?
   (b) If $a_n < b_n$ for all n, what can you say about Σa_n? Why?

2. Suppose Σa_n and Σb_n are series with positive terms and Σb_n is known to be divergent.
   (a) If $a_n > b_n$ for all n, what can you say about Σa_n? Why?
   (b) If $a_n < b_n$ for all n, what can you say about Σa_n? Why?

3–32 Determine whether the series converges or diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+8}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$

10. $\displaystyle\sum_{k=1}^{\infty} \frac{k\sin^2 k}{1+k^3}$

11. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$

12. $\displaystyle\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{e^n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n}$

22. Σ

23. $\displaystyle\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$

24. Σ

25. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1}$

26. Σ

27. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$

28. Σ

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

30. Σ

31. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

32. Σ

33–36 Use the sum of the first 10 terms to ... the series. Estimate the error.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5+n^5}$

34.

35. $\displaystyle\sum_{n=1}^{\infty} 5^{-n}\cos^2 n$

36.

37. The meaning of the decimal represent... $0.d_1d_2d_3\ldots$ (where the digit d_i is one ... $2, \ldots, 9$) is that

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots$$

Show that this series always converge...