Exam #3 Review Packet #65-83

65. $\lim_{x\to 0} \frac{\sin(3x)-3x}{x-\arctan x} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{3\cos(3x)-3}{1-\frac{1}{1+x^2}} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{-9\sin(3x)}{\frac{2x}{(1+x^2)^2}}$ 0/0

$-(1+x^2)^{-1} \to +(1+x^2)^{-2}(2x)$

$\underset{L'H}{=} \lim_{x\to 0} \frac{-27\cos(3x)}{\frac{(1+x^2)^2(2)-2x(2)(1+x^2)(2x)}{(1+x^2)^4}} = \boxed{\frac{-27}{2}}$

Series

$\lim_{x\to 0} \frac{\sin(3x)-3x}{x-\arctan x} = \lim_{x\to 0} \frac{\cancel{(3x)} - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \dots - \cancel{3x}}{\cancel{x} - (\cancel{x} - x^3/3 + x^5/5 - \dots)}$

$= \lim_{x\to 0} \frac{-\frac{27x^3}{3!} + \frac{3^5x^5}{5!} - \dots}{\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \dots} \frac{(1/x^3)}{\left(\frac{1}{x^3}\right)}$

$= \lim_{x\to 0} \frac{-\frac{27}{6} + \frac{3^5}{5}x^2 - \dots \to 0}{\frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \dots \to 0} = \frac{\frac{-27}{6}}{1/3}{}^{3} = \boxed{\frac{-27}{2}}$ Match!

$-(1+x^2)$

L'H

66. $\lim_{x\to 0} \frac{xe^x - \arctan x}{\ln(1+3x) - 3x} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{xe^x + e^x - \frac{1}{1+x^2}}{\frac{3}{1+3x} - 3}$ $0/0$

$3(1+3x)^{-1}$

$$\underset{L'H}{=} \lim_{x\to 0} \frac{\cancel{(xe^x+e^x)}^{\to 0} + \overset{1}{e^x} + \cancel{\frac{(2x)}{(1+x^2)^2}}^{\to 0}}{\cancel{\frac{-9}{(1+3x)^2}}^{\to -9}} = \boxed{\frac{-2}{9}}$$

Limits

$$\lim_{x\to 0} \frac{xe^x - \arctan x}{\ln(1+3x) - 3x} = \lim_{x\to 0} \frac{x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)}{\cancel{(3x)} - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \dots \cancel{-3x}}$$

$$= \lim_{x\to 0} \frac{\cancel{x} + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \dots \cancel{-x} + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \dots}{-\frac{9x^2}{2} + \frac{27x^3}{3} - \dots}$$

$$= \lim_{x\to 0} \frac{x^2 + \frac{5x^3}{6} + \frac{x^4}{3!} - \dots}{-\frac{9x^2}{2} + 9x^3 - \dots} \quad \frac{(1/x^2)}{(1/x^2)}$$

$$= \lim_{x\to 0} \frac{1 + \cancel{\frac{5x}{6} + \frac{x^2}{3!} + \dots}^{\to 0}}{-9/2 + \cancel{9x + \dots}^{\to 0}} = \frac{1}{(-9/2)} = \boxed{\frac{-2}{9}}$$ Match!

67. $\sum_{n=1}^{\infty} \frac{6^n}{n!} \rightarrow L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{6^{n+1}}{(n+1)!}}{\frac{6^n}{n!}}\right| = \lim_{n\to\infty} \frac{\cancel{6^{n+1}}^{\,6}}{\cancel{6^n}} \cdot \frac{\cancel{n!}}{\underbrace{(n+1)!}_{(n+1)\cancel{n!}}} = \lim_{n\to\infty} \frac{6}{n+1} \xrightarrow{} 0 = 0 < 1$

Series Converges by Ratio Test

$\Rightarrow \lim_{n\to\infty} \frac{6^n}{n!} = 0$ because otherwise

$\underline{\underline{\text{if}}}$ $\lim_{n\to\infty} \frac{6^n}{n!} \neq 0$ then the Series would Diverge by nTDT which would contradict the Ratio Test above.

68. $\sum_{n=1}^{\infty} \frac{n^n \cdot n!}{(3n)!} \rightarrow L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}(n+1)!}{[3(n+1)]!}}{\frac{n^n \cdot n!}{(3n)!}}\right| = \lim_{n\to\infty} \frac{\overbrace{(n+1)^{n+1}}^{(n+1)^n \cdot (n+1)}}{n^n} \cdot \frac{\overbrace{(n+1)!}^{(n+1)n!}}{\cancel{n!}} \cdot \frac{\cancel{(3n)!}}{\underbrace{(3n+3)!}_{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}}$

$= \lim_{n\to\infty} \underbrace{\frac{(n+1)^n}{n^n}}_{\to e} \frac{\cancel{(n+1)}\cancel{(n+1)}}{\underbrace{(3n+3)}_{3(n+1)}(3n+2)(3n+1)} = \lim_{n\to\infty} \frac{e}{9} \cdot \left(\frac{1}{3n+1}\right) \to 0 = 0 < 1$

Series Converges by RT

$\Rightarrow \lim_{n\to\infty} \frac{n^n \cdot n!}{(3n)!} = 0$ because otherwise

$\underline{\underline{\text{if}}}$ $\lim_{n\to\infty} \frac{n^n \cdot n!}{(3n)!} \neq 0$ then the Series would Diverge by nTDT which would contradict the Ratio Test above.

69. $\int \cos(x^2) - 1 + \frac{x^4}{2}\, dx = \int \underbrace{\cancel{1}}_{n=0} - \underbrace{\cancel{\frac{(x^2)^2}{2!}}}_{n=1,\ \frac{x^4}{2}} + \underbrace{\frac{(x^2)^4}{4!}}_{n=2} - \cdots \cancel{-1} + \cancel{\frac{x^4}{2}}\, dx$

$= \int \sum_{n=2}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}\, dx = \int \sum_{n=2}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\, dx = \boxed{\sum_{n=2}^{\infty} \frac{(-1)^n\, x^{4n+1}}{(2n)!\,(4n+1)} + C}$

70. $\displaystyle\int \sin(x^2) - x^2\,dx = \int \overset{n=0}{\cancel{(x^2)}} - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots \cancel{-x^2}\,dx$

$$= \int \sum_{n=1}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\,dx = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!(4n+3)} + C}$$

71. $\displaystyle\int 1 - \cos(x^2)\,dx = \int \cancel{1} - \Big(\overset{n=0}{\cancel{1}} - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \dots\Big)\,dx$

extra minus

$$= \int -\sum_{n=1}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n}}{(2n)!}\,dx = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+1}}{(2n)!(4n+1)} + C}$$

72. $\displaystyle\int 1 - x^2 - e^{-x^2}\,dx = \int \cancel{1 - x^2} - \Big(\overset{n=0}{\cancel{1}} + \overset{n=1}{\cancel{(-x^2)}} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots\Big)\,dx$

$-1 + x^2 \dots$

extra minus

$$= \int -\sum_{n=2}^{\infty} \frac{(-x^2)^n}{n!}\,dx = \int -\sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \int \sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n!}\,dx = \boxed{\sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{n!(2n+1)} + C}$$

73. $\displaystyle\int \arctan(2x) - 2x + \frac{8x^3}{3}\,dx = \int \overset{n=0}{\cancel{2x}} - \overset{n=1}{\cancel{\frac{(2x)^3}{3}}} + \frac{(2x)^5}{5} - \frac{(2x)^7}{7} + \dots \cancel{-2x} \cancel{+\frac{8x^3}{3}}\,dx$

$$= \int \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}\,dx = \int \sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}\,dx$$

$$= \boxed{\sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+2}}{(2n+1)(2n+2)} + C}$$

74. (a) $f(x) = x^5\sin(x^3) = x^5 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} = x^5 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+8}}{(2n+1)!}$

$= \frac{x^8}{1} \underset{0x^9}{\uparrow} - \frac{x^{14}}{3!} + \ldots$

Maclaurin Series

(b) $f(0) + f'(0)x + \cdots + \frac{f^{(8)}(0)}{8!}x^8 + \frac{f^{(9)}(0)}{9!}x^9 + \ldots$

$$\begin{array}{r} 5040 \\ \times\ 8 \\ \hline 40320 \end{array}$$

Match Coefficients $\frac{f^{(8)}(0)}{8!} = 1 \Rightarrow f^{(8)}(0) = 8! = 8 \cdot \overset{5040}{7!} = \boxed{40{,}320}$

$\frac{f^{(9)}(0)}{9!} = 0 \Rightarrow f^{(9)}(0) = \boxed{0}$ No x^9 Term $\Rightarrow$ Coefficient $= 0$

75. (a) $f(x) = xe^{-x^7} = x \sum_{n=0}^{\infty} \frac{(-x^7)^n}{n!} = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{7n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{7n+1}}{n!}$

$= x - x^8 + \frac{x^{15}}{2!} \underset{0x^{21}}{\uparrow} - \frac{x^{22}}{3!} + \frac{x^{29}}{4!} - \ldots$

Maclaurin Series

(b) $f(0) + f'(0)x + \cdots + \frac{f^{(20)}(0)}{(20)!}x^{20} + \frac{f^{(21)}(0)}{(21)!}x^{21} + \frac{f^{(22)}(0)}{(22)!}x^{22} + \cdots$

Match Coefficients $\frac{f^{(21)}(0)}{(21)!} = 0 \Rightarrow f^{(21)}(0) = \boxed{0}$

$\frac{f^{(22)}(0)}{(22)!} = -\frac{1}{3!} \Rightarrow f^{(22)}(0) = \boxed{\frac{-(22)!}{3!}}$

76. (a) $f(x) = x^5 \ln(1+3x) = x^5 \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{n+1}}{n+1} = x^5 \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} x^{n+6}}{n+1}$

$= \frac{3x^6}{1} - \frac{3^2 x^7}{2} + \frac{3^3 x^8}{3} - \frac{3^4 x^9}{4} + \cdots$

Maclaurin Series

(b) $f(0) + f'(0)x + \cdots + \frac{f^{(7)}(0)}{7!}x^7 + \frac{f^{(8)}(0)}{8!}x^8 + \frac{f^{(9)}(0)}{9!}x^9 + \cdots$

Match Coefficients $\frac{f^{(7)}(0)}{7!} = -\frac{9}{2} \Rightarrow f^{(7)}(0) = \boxed{\frac{-9 \cdot 7!}{2}}$

$\frac{f^{(9)}(0)}{9!} = -\frac{81}{4} \Rightarrow f^{(9)}(0) = \boxed{\frac{-81 \cdot 9!}{4}}$

77. (a) $f(x) = x\arctan(x^2) = x \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1} = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{2n+1}$

$= \frac{x^3}{1} - \frac{x^7}{3} + \frac{x^{11}}{5} - \cdots$

$0x^8$

Maclaurin Series

(b) $f(0) + f'(0)x + \cdots + \frac{f^{(7)}(0)}{7!}x^7 + \frac{f^{(8)}(0)}{8!}x^8 + \cdots$

$7! = 7 \cdot \overset{720}{6!} = 5040$

$5040 \div 3 = 1680$

Match Coefficients $\frac{f^{(7)}(0)}{7!} = -\frac{1}{3} \Rightarrow f^{(7)}(0) = -\frac{7!}{3} = -\frac{5040}{3} = \boxed{-1680}$

$\frac{f^{(8)}(0)}{8!} = 0 \Rightarrow f^{(8)}(0) = \boxed{0}$

78.(a) Chart Method (by Definition)

$f(x)=\cosh x \quad f(0)=\cosh(0)=1$

$f'(x)=\sinh x \quad f'(0)=\sinh(0)=0$

$f''(x)=\cosh x \quad f''(0)=\cosh(0)=1$

$f'''(x)=\sinh x \quad f'''(0)=\sinh 0=1$

$\vdots$

$$\overset{1}{f(0)}+\overset{0}{f'(0)}x+\frac{\overset{1}{f''(0)}x^2}{2!}+\frac{\overset{0}{f'''(0)}x^3}{3!}+\cdots$$

$$=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=\boxed{\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}}$$

(b) $$\cosh x=\frac{e^x+e^{-x}}{2}=\frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right]$$

$$=\frac{1}{2}\left[\left(1+\cancel{x}+\frac{x^2}{2!}+\cancel{\frac{x^3}{3!}}+\frac{x^4}{4!}+\cdots\right)+\left(1-\cancel{x}+\frac{x^2}{2!}-\cancel{\frac{x^3}{3!}}+\frac{x^4}{4!}-\cdots\right)\right]$$

$$=\frac{1}{\cancel{2}}\left[\cancel{2}+\cancel{2}\cdot\frac{x^2}{2!}+\cancel{2}\cdot\frac{x^4}{4!}+\cdots\right]$$

$$=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=\boxed{\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}}$$

(c) $$\sinh x=\frac{e^x-e^{-x}}{2}=\frac{1}{2}\left[\cancel{1}+x+\cancel{\frac{x^2}{2!}}+\frac{x^3}{3!}+\cdots-\left(\cancel{1}-x+\cancel{\frac{x^2}{2!}}-\frac{x^3}{3!}+\cdots\right)\right]$$

$$=\frac{1}{\cancel{2}}\left[\cancel{2}x+\cancel{2}\cdot\frac{x^3}{3!}+\cancel{2}\cdot\frac{x^5}{5!}+\cdots\right]=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh x=\frac{d}{dx}\left[\sinh x\right]=\frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\right]=\sum_{n=0}^{\infty}\frac{\cancel{(2n+1)}x^{2n}}{\cancel{(2n+1)!}}=\boxed{\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}}$$

$(2n+1)(2n)!$

78 (d) $f(x)=\cosh(3x^2)=\sum_{n=0}^{\infty}\frac{(3x^2)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{3^{2n}x^{4n}}{(2n)!}$

$$\overset{n=0}{=1}+\overset{n=1}{\frac{9x^4}{2!}}+\overset{n=2}{\frac{81x^8}{4!}}+\dots$$

(e) $f(0)+f'(0)x+\dots+\underbrace{\frac{f^{(7)}(0)}{7!}x^7}_{0x^7}+\frac{f^{(8)}(0)}{8!}x^8+\dots$

Match coefficients $\frac{f^{(7)}(0)}{7!}=0\Rightarrow f^{(7)}(0)=\boxed{0}$

$$\frac{f^{(8)}(0)}{8!}=\frac{81}{4!}\Rightarrow f^{(8)}(0)=\boxed{\frac{(81)\cdot 8!}{4!}}$$

79. $\sum_{n=0}^{\infty}\frac{n}{4^n}$ looks like $\sum_{n=0}^{\infty}nx^n$ with $x=\frac{1}{4}$ $\qquad (1-x)^{-1}\to -(1-x)^{-2}(-1)$

$$\sum_{n=0}^{\infty}nx^n=x\sum_{n=0}^{\infty}n\cdot x^{n-1}=x\frac{d}{dx}\left[\overset{\frac{1}{1-x}}{\sum_{n=0}^{\infty}x^n}\right]=x\frac{d}{dx}\left[\frac{1}{1-x}\right]=x\left[\frac{1}{(1-x)^2}\right]=\frac{x}{(1-x)^2}$$

Plug in $x=\frac{1}{4}\Rightarrow\sum_{n=0}^{\infty}\frac{n}{4^n}=\frac{\frac{1}{4}}{(1-\frac{1}{4})^2}=\frac{\frac{1}{4}}{(\frac{3}{4})^2}=\frac{\frac{1}{4}}{\frac{9}{16}}\cdot\frac{16}{9}=\boxed{\frac{4}{9}}$

80. $\sum_{n=0}^{\infty}\frac{n^2}{4^n}$ looks like $\sum_{n=0}^{\infty}n^2x^n$ with $x=\frac{1}{4}$

$$\sum_{n=0}^{\infty}n^2x^n=x\sum_{n=0}^{\infty}n\cdot nx^{n-1}=x\frac{d}{dx}\sum_{n=0}^{\infty}n\cdot x^n=x\frac{d}{dx}\left[x\sum_{n=0}^{\infty}nx^{n-1}\right]=x\frac{d}{dx}\left[x\frac{d}{dx}\overset{\frac{1}{1-x}}{\sum_{n=0}^{\infty}x^n}\right]$$

$$=x\frac{d}{dx}\left[x\frac{d}{dx}\left(\frac{1}{1-x}\right)\right]=x\frac{d}{dx}\left[x\frac{1}{(1-x)^2}\right]=x\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right]=x\left[\frac{(1-x)^2(1)-x\cdot 2(1-x)(-1)}{(1-x)^4}\right]$$

$$=x\left[\frac{\cancel{(1-x)}((1-x)+2x)}{(1-x)^{\cancel{4}}}\right]=\frac{x(x+1)}{(1-x)^3}\Rightarrow\sum_{n=0}^{\infty}\frac{n^2}{4^n}=\frac{\frac{1}{4}(\frac{1}{4}+1)}{(1-\frac{1}{4})^3}=\frac{\frac{1}{4}(\frac{5}{4})}{(\frac{3}{4})^3}=\frac{5}{\cancel{16}}\cdot\frac{\cancel{64}^{4}}{27}=\boxed{\frac{20}{27}}$$

81. $\sum_{n=0}^{\infty} \frac{n(\ln 3)^n}{n!}$ looks like $\sum_{n=0}^{\infty} \frac{nx^n}{n!}$ with $x = \ln 3$

$$\sum_{n=0}^{\infty} \frac{nx^n}{n!} = x\sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!} = x\frac{d}{dx}\overbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}^{e^x} = x\frac{d}{dx}(e^x) = xe^x$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{n(\ln 3)^n}{n!} = \ln 3 \cdot \cancel{e^{\ln 3}} = \boxed{3 \cdot \ln 3} = \ln(27)$$

82. $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}} = \sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$$

$$= \sqrt{3}\, \overbrace{\arctan\left(\frac{1}{\sqrt{3}}\right)}^{\pi/6} = \boxed{\frac{\sqrt{3}\pi}{6}}$$

83. $$\sum_{n=2}^{\infty} \frac{(-1)^n}{2n+1} = \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \underbrace{\overbrace{\arctan(1)}^{\pi/4}}_{1 - 1/3 + 1/5 - 1/7 + \dots} \overbrace{- 1 + \frac{1}{3}}^{-2/3} = \boxed{\frac{\pi}{4} - \frac{2}{3}}$$