Fall 2020

Exam #3 Final Answers

1(a). $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (3x+4)^n}{(n+7)^2 \cdot 8^n}$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)}^{n+1}(3x+4)^{n+1}}{(n+8)^2 \cdot 8^{n+1}}}{\dfrac{\cancel{(-1)}^n (3x+4)^n}{(n+7)^2 \cdot 8^n}} \right|$$

$$= \lim_{n\to\infty} \left|\frac{(3x+4)^{n+1}}{(3x+4)^n}\right| \cdot \frac{(n+7)^2}{(n+8)^2} \cdot \frac{8^n}{8^{n+1}}$$

$$= \lim_{n\to\infty} |3x+4| \cdot \left[\frac{n+7}{n+8}\right]^2 \cdot \frac{1}{8}$$

$$= \frac{|3x+4|}{8} < 1$$

Converges by Ratio Test when

$|3x+4| < 8$

$-8 < 3x+4 < 8$

$-12 < 3x < 4$

$-4 < x < 4/3$

Manually Check Convergence at Endpoints

Take $x = 4/3$ Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n \left[\cancel{3}\left(\frac{4}{\cancel{3}}\right)+4\right]^n}{(n+7)^2 \cdot 8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{8^n}}{(n+7)^2 \cdot \cancel{8^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+7)^2}$$

Converges by AST (or ACT with $\sum_{n=0}^{\infty} \frac{1}{(n+7)^2}$)

① $b_n = \dfrac{1}{(n+7)^2} > 0$

② $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(n+7)^2} = 0$

③ $b_{n+1} = \dfrac{1}{(n+8)^2} \le \dfrac{1}{(n+7)^2} = b_n$ Terms Decreasing

Take $x=-4$ Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n\left[3(-4)+4\right]^n}{(n+7)^2 \cdot 8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-8)^n}{(n+7)^2 \cdot \cancel{8^n}} = \sum_{n=1}^{\infty} \frac{\cancel{(-1)^{2n}}^{\,1}}{(n+7)^2}$$

$(-1)^n \cancel{8^n}$ Even power

$$= \sum_{n=1}^{\infty} \frac{1}{(n+7)^2} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$$ Converges, p-Series $p=2>1$

CT or LCT work. Bound Terms

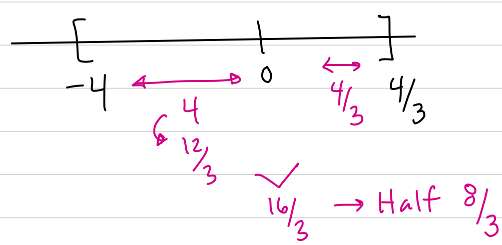
$$\frac{1}{(n+7)^2} \le \frac{1}{n^2}$$ $\Rightarrow$ Series also Converges by CT

OR LCT

$$\lim_{n\to\infty} \frac{\frac{1}{(n+7)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+7)^2} = \lim_{n\to\infty} \left(\frac{n}{n+7}\right)^2 = 1$$ Finite, Non-Zero

$\Rightarrow$ Series Converges by LCT

$$\boxed{I = \left[-4, \tfrac{4}{3}\right] \quad R = \tfrac{8}{3}}$$



1(b). $\sum_{n=1}^{\infty} (2n)!\,(\ln n)\,(x-7)^n$

Ratio Test

$(2n+2)(2n+1)\cancel{(2n)!}$

$(2n+2)!$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{[2(n+1)]!\ \ln(n+1)\ (x-7)^{n+1}}{\cancel{(2n)!}\ \ln n\ (x-7)^n} \right|$$

$(*)$ $\frac{\ln(n+1)}{\ln n} \to 1$

$$= \lim_{n\to\infty} (2n+2)(2n+1)\,|x-7| = \infty > 1$$ Diverges by R.T. for all x unless

$x=7$ (when $L=0<1$)

$$\boxed{I = \{7\} \quad R = 0}$$

(*) $\displaystyle \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{1}{1} = 1$

1(c). $\displaystyle \sum_{n=1}^{\infty} \frac{x^{3n-1}}{n^n}$

Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{3(n+1)-1}}{(n+1)^{n+1}}}{\frac{x^{3n-1}}{n^n}} \right| = \lim_{n\to\infty} \underbrace{\left| \frac{x^{3n+2}}{x^{3n-1}} \right|}_{|x|^3} \cdot \underbrace{\frac{n^n}{(n+1)^{n+1}}}_{\to \frac{1}{e}}$$

$(n+1)^{n+1} = (n+1)^n (n+1)$

$$= \lim_{n\to\infty} \cancel{\frac{|x|^3}{e\,(n+1)}} \to 0 = 0 < 1$$

$(n+1) \to \infty$

Converges by R.T. for all Real Numbers

$$\boxed{\begin{array}{l} I = (-\infty, \infty) \\ R = \infty \end{array}}$$

2(a). $\displaystyle x^3 \arctan(7x) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (7x)^{2n+1}}{2n+1} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n\, \overset{\text{split}}{7^{2n+1}\, x^{2n+1}}}{2n+1}$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n\, 7^{2n+1}\, x^{2n+4}}{2n+1}}$$

Need $|7x| < 1$

$\Rightarrow |x| < 1/7 \Rightarrow \boxed{R = \frac{1}{7}}$

2(b). $\displaystyle \frac{d}{dx}\left[x^3 \arctan(7x)\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n\, 7^{2n+1}\, x^{2n+4}}{2n+1}\right]$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n\, 7^{2n+1}\, (2n+4) x^{2n+3}}{2n+1}}$$

2(c). $\displaystyle \int x^3 \arctan(7x)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n\, 7^{2n+1}\, x^{2n+4}}{2n+1}\,dx$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n\, 7^{2n+1}\, x^{2n+5}}{(2n+1)\,(2n+5)} + C}$$

3. $\int_0^1 x^2 e^{-x^3}dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} dx$

$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n!\,(3n+3)} \Big|_0^1$

$n=0 \quad n=1 \quad n=2$

$= \frac{x^3}{3} - \frac{x^6}{6} + \frac{x^9}{2!\cdot 9} - \frac{x^{12}}{3!\cdot(12)} + \cdots \Big|_0^1$

$= \frac{1}{3} - \frac{1}{6} + \frac{1}{18} - \frac{1}{72} + \cdots - \cancel{(0-0+0-\cdots)}$ 0

$\approx \frac{1}{3} - \frac{1}{6} + \frac{1}{18} = \frac{6}{18} - \frac{3}{18} + \frac{1}{18} = \frac{4}{18} = \boxed{\frac{2}{9}}$ Estimate

Using ASET, we can estimate the full sum using only the first three terms with error at most |first neglected term|

$\frac{1}{72} < \frac{1}{50}$ as desired.

4 (a) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/3)^{2n}}{(2n+1)!} \frac{\pi/3}{\pi/3} = \frac{3}{\pi} \sin\left(\frac{\pi}{3}\right) = \boxed{\frac{3\sqrt{3}}{2\pi}}$

3^{2n} $\quad (\pi/3)^{2n+1}$ $\quad \sqrt{3}/2$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(b). $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{n+1} (\ln 9)^n}{n!} = -2 \sum_{n=0}^{\infty} \frac{(-2\ln 9)^n}{n!} = -2e^{-2\ln 9}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad = -2\cancel{e^{\ln}}(9^{-2}) = \frac{-2}{9^2} = \boxed{\frac{-2}{81}}$

(c). $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \pi^{2n+1}}{9\,(2n)!} = -\frac{\pi}{9} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = -\frac{\pi}{9} \cdot \cancel{\cos \pi} = \boxed{\frac{\pi}{9}}$

constant $\quad -1$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

(d). $-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\cdots = -\left(1-1/3+1/5-\frac{1}{7}+1/9-\cdots\right) = -\cancel{\arctan}^{\pi/4} 1 = \boxed{-\frac{\pi}{4}}$

$\arctan x = x - \frac{x^3}{3}+\frac{x^5}{5}-\cdots$ or $\arctan(-1) = \frac{-\pi}{4}$ also works

(e). $\sum_{n=0}^{\infty} \frac{1}{3!\,\pi^n} = \frac{1}{3!}\sum_{n=0}^{\infty}\left(\frac{1}{\pi}\right)^n = \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{1}{\pi}\right)^n$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$= \frac{1}{6}\left[\frac{1}{1-1/\pi}\right] = \boxed{\frac{1}{6}\left[\frac{\pi}{\pi-1}\right]}$

$\frac{\pi-1}{\pi}$

(f). $\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots = \ln 2 - 1 + 1/2 = \boxed{(\ln 2)-\frac{1}{2}}$

$\ln(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

$\underbrace{\ln(1+1)}_{\ln 2} = 1 - \underbrace{\frac{1}{2}}_{1/2 \text{ missing}} + \frac{1}{3} - 1/4 + \cdots$

5. (a). Chart Method / Definition of Maclaurin Series

$f(x) = \cos x \qquad f(0) = \cos 0 = 1$

$f'(x) = -\sin x \qquad f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \qquad f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \qquad f'''(0) = \sin 0 = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = \cos 0 = 1$

$\vdots \qquad \vdots$

Maclaurin Series

$\cancel{f(0)}^{1} + \cancel{f'(0)}^{0}x + \frac{\cancel{f''(0)}^{-1}}{2!}x^2 + \frac{\cancel{f'''(0)}^{0}}{3!}x^3 + \frac{\cancel{f^{(4)}(0)}^{1}}{4!}x^4 + \cdots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}}$

(b). Differentiation

$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n \cancel{(2n+1)}\, x^{2n}}{\cancel{(2n+1)!}} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}}$

$\cancel{(2n+1)}\ (2n)!$

Done in class

OR, expand

$$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= 1 - \frac{\cancel{3}}{\cancel{3!}}_{2!}x^2 + \frac{\cancel{5}}{\cancel{5!}}_{4!}x^4 - \frac{\cancel{7}}{\cancel{7!}}_{6!}x^6 + \cdots = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}}$$

(c) Integration

$$\cos x = \int -\sin x\,dx = \int -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)dx$$

$$= \int -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots\,dx$$

$$= -\frac{x^2}{2} + \underbrace{\frac{x^4}{3!\cdot 4}}_{4!} - \underbrace{\frac{x^6}{5!\cdot 6}}_{6!} + \underbrace{\frac{x^8}{7!\cdot 8}}_{8!} - \cdots + C$$

$$= -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + C$$

Test $x=0$ to Solve for $+C$

$$\cancel{\cos 0}^{\,1} = -0 + 0 - 0 + \cdots + C \Rightarrow C = 1$$

Finally, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}}$

6. $x = e^t + \frac{1}{1+e^t}$ $\qquad$ $y = 2\ln(1+e^t)$

$\frac{dx}{dt} = e^t - \frac{e^t}{(1+e^t)^2}$ $\qquad$ $\frac{dy}{dt} = \frac{2e^t}{1+e^t}$

(a). $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2e^t}{1+e^t}}{e^t - \frac{e^t}{(1+e^t)^2}}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{\frac{2e^0}{1+e^0}}{e^0 - \frac{e^0}{(1+e^0)^2}} = \frac{\frac{\cancel{2}}{\cancel{1+1}}}{1 - \frac{1}{(1+1)^2}} = \frac{1}{1-\frac{1}{4}} = \frac{1}{\left(\frac{3}{4}\right)} = \boxed{\frac{4}{3}}$$ Match!

(b). Arclength

$$L = \int_0^{\ln 3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\ln 3} \sqrt{\left[e^t - \frac{e^t}{(1+e^t)^2}\right]^2 + \left[\frac{2e^t}{1+e^t}\right]^2}\, dt$$

$$= \int_0^{\ln 3} \sqrt{e^{2t} - \frac{2e^{2t}}{(1+e^t)^2} + \frac{e^{2t}}{(1+e^t)^4} + \frac{4e^{2t}}{(1+e^t)^2}}\, dt$$

$$= \int_0^{\ln 3} \sqrt{e^{2t} + \frac{2e^{2t}}{(1+e^t)^2} + \frac{e^{2t}}{(1+e^t)^4}}\, dt$$

$$= \int_0^{\ln 3} \sqrt{\left(e^t + \frac{e^t}{(1+e^t)^2}\right)^2}\, dt$$

$$= \int_0^{\ln 3} e^t + \frac{e^t}{(1+e^t)^2}\, dt = e^t - \frac{1}{(1+e^t)}\Bigg|_0^{\ln 3}$$

reverse $\frac{dx}{dt}$ above

$$= \cancelto{3}{e^{\ln 3}} - \frac{1}{1+\cancelto{3}{e^{\ln 3}}} - \left(\cancelto{1}{e^0} - \frac{1}{1+\cancelto{1}{e^0}}\right)$$

$$= 3 - \tfrac{1}{4} - 1 + \tfrac{1}{2} = 2 + \tfrac{1}{4} = \boxed{\frac{9}{4}}$$