Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 121

Midterm Exam #3

December 6, 2017

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{3\ln 3}$, $\arctan\sqrt{3}$ or $\cosh(\ln 3)$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		20
2		12
3		10
4		8
5		20
6		18
7		12
Total		100

1. [20 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x-4)^n}{5^n\,\sqrt{n}}$

Converges by R.T. when

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}\,(3x-4)^{n+1}}{5^{n+1}\sqrt{n+1}}}{\dfrac{\cancel{(-1)^{n}}\,(3x-4)^{n}}{5^{n}\sqrt{n}}}\right| = \lim_{n\to\infty}\left|\frac{(3x-4)^{n+1}}{(3x-4)^n}\right|\cdot\frac{5^n}{5^{n+1}}\underbrace{\sqrt{\frac{n}{n+1}}}_{1} = \frac{|3x-4|}{5} < 1$$

$|3x-4| < 5$

$$\begin{aligned} -5 &< 3x-4 < 5 \\ +4 \quad & \quad\; +4 \quad +4 \\ -1 &< 3x < 9 \\ -\tfrac{1}{3} &< x < 3 \end{aligned}$$

Manually Check Convergence at Endpoints

$x=3$ O.S. becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(9-4)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n\,\cancel{5^n}}{\cancel{5^n}\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

Converges by AST

① $b_n = \dfrac{1}{\sqrt{n}} > 0$

② $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

③ $b_{n+1} < b_n$

$\dfrac{1}{\sqrt{n+1}} < \dfrac{1}{\sqrt{n}}$ ✓

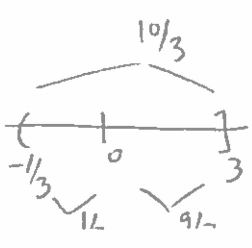
OR $f(x) = \dfrac{1}{\sqrt{x}} \Rightarrow f'(x) = \dfrac{-1}{x^{3/2}} < 0$

$x=-\frac{1}{3}$ O.S. becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(-1-4)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{\overbrace{(-1)^n(-1)^n}^{(-1)^{2n}}\,\cancel{5^n}}{\cancel{5^n}\sqrt{n}} = \sum\frac{1}{\sqrt{n}}$

Diverges p-series
$p = \frac{1}{2} < 1$

$I = \boxed{\left(-\frac{1}{3}, 3\right]}$

$R = \boxed{\frac{5}{3}}$ Half Total Length of $10/3$



1. (Continued) Find the **Interval** and **Radius** of Convergence for each of the following power series. Analyze carefully and with full justification.

(b) $\sum_{n=1}^{\infty} n!\,(x-6)^n$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)!\,(x-6)^{n+1}}{n!\,(x-6)^n}\right|=\lim_{n\to\infty} n|x-6|=\infty>1$$

Diverges by R.T. for all $\mathbb{R}$ unless $x=6$

$$\boxed{\begin{array}{c} I=\{6\} \\ R=0 \end{array}}$$

(c) $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{[2(n+1)]!}}{\frac{x^n}{(2n)!}}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|\frac{\cancel{(2n)!}}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)\cancel{(2n)!}}}=\lim_{n\to\infty}\frac{\overset{0}{\cancel{|x|}}}{(2n+2)(2n+1)}=0<1$$

Converges by R.T. for all x in $\mathbb{R}$.

$$\boxed{\begin{array}{l} I=\mathbb{R} \text{ or } (-\infty,\infty) \\ R=\infty \end{array}}$$

2. [12 Points] Find the **MacLaurin series** representation for each of the following functions. **State** the Radius of Convergence for each series. Your answer should be in sigma notation $\sum_{n=0}^{\infty}$. Simplify.

(a) $f(x) = x^3 \ln(1+2x)$ $= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{n+1}}{n+1}$

$$= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{n+1}}{n+1}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{n+4}}{n+1}}$$

Need $|2x| < 1$

$\Rightarrow |x| < 1/2$

$\boxed{R = 1/2}$

(b) $f(x) = 1 - \cos(x^2)$

deletes $n=0$ term.

distribute minus

$$= 1 - \sum_{n=0}^{\infty} \frac{(-1)^n \overbrace{(x^2)^{2n}}^{x^{4n}}}{(2n)!} = 1 - \left(\cancel{1} - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots\right)$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

$$= \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n}}{(2n)!}}$$

$\boxed{R = \infty}$

OR $\sum_{n=0}^{\infty} \frac{\overbrace{(-1)^{n+2}}^{(-1)^n} x^{4n+4}}{(2n+2)!}$

reindex.

3. [10 Points] Use the MacLaurin Series representation for $f(x) = x\sin(x^2)$ to

Estimate $\int_0^1 x\sin(x^2)\,dx$ with error less than $\frac{1}{10}$.

Justify in words that your error is indeed less than $\frac{1}{10}$.

$$\int_0^1 x\sin(x^2)dx = \int_0^1 x\sum_{n=0}^{\infty}\frac{(-1)^n(x^2)^{2n+1}}{(2n+1)!}dx = \int_0^1 x\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!}dx$$

$$= \int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(2n+1)!}dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+4}}{(2n+1)!\,(4n+4)}\Bigg|_0^1$$

$$= \frac{x^4}{1\cdot 4} - \frac{x^8}{3!\cdot 8} + \frac{x^{12}}{5!\cdot(12)} - \dots \Bigg|_0^1$$

$$= \frac{1}{4} - \frac{1}{48} + \dots - \cancel{(0-0+0-\dots)}\ 0$$

$$\approx \boxed{\frac{1}{4}} \leftarrow \text{Estimate.}$$

Using ASET, we can estimate the full Alternating Sum using only the 1st Term with error at most the

$|\text{First Neglected Term}| = \frac{1}{48} < \frac{1}{10}$ as desired

4. [8 Points] Estimate $\arctan\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$. Justify in words that your error is indeed less than $\frac{1}{100}$.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

$$\begin{array}{r} 32 \\ \underline{5} \\ 160 \end{array}$$

$$\arctan\left(\tfrac{1}{2}\right) = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \ldots$$

$$= \underbrace{\frac{1}{2} - \frac{1}{24}} + \frac{1}{160} - \ldots$$

$$\approx \frac{1}{2} - \frac{1}{24} = \frac{12}{24} - \frac{1}{24} = \boxed{\frac{11}{24}} \leftarrow \text{Estimate}$$

Using ASET, we can estimate the full Alternating Sum using only the First Two Terms with error at most the $|\text{First Neglected Term}| = \frac{1}{160} < \frac{1}{100}$ as desired.

5. [20 Points] Find the **sum** for each of the following series. Simplify, if possible.

(a) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$ $= \arctan(1) = \boxed{\frac{\pi}{4}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \, \pi^{2n}}{36^n \, (2n+1)!}$ $= \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/6)^{2n}}{(2n+1)!} \cdot \frac{(\pi/6)}{(\pi/6)}$

$= \displaystyle\frac{6}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/6)^{2n+1}}{(2n+1)!} = \frac{6}{\pi} \underbrace{\sin(\pi/6)}_{1/2} = \frac{6}{\pi}\left(\frac{1}{2}\right) = \boxed{\frac{3}{\pi}}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \, (\ln 27)^n}{3^{n+1} \, n!}$

$= \displaystyle\frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n (\ln 27)^n}{3^n \, n!} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{\left(\frac{-\ln 27}{3}\right)^n}{n!} = \frac{1}{3} e^{\frac{-\ln 27}{3}} = \frac{1}{3} \cancel{e}^{\cancel{\ln}(27^{-1/3})} = \frac{1}{3} \cdot \frac{1}{\underbrace{\sqrt[3]{27}}_{3}}$

$= \boxed{\frac{1}{9}}$

(d) $1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} + \frac{e^4}{4!} - \frac{e^5}{5!} + \ldots$ $= \boxed{e^{-e}}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(e) $\frac{1}{e} - \frac{1}{2e^2} + \frac{1}{3e^3} - \frac{1}{4e^4} + \frac{1}{5e^5} - \ldots$ $= \boxed{\ln\left(1 + \frac{1}{e}\right)} \overset{\text{OR}}{=} \boxed{\ln\left(\frac{e+1}{e}\right)} \overset{\text{OR}}{=} \ln(e+1) - \cancel{\ln} e$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

$= \boxed{\ln(e+1) - 1}$

(f) $-\frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \ldots$ $= \underbrace{\cancel{\cos} \pi}_{-1} - 1 = \boxed{-2}$

missing $n=0$ term

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\cos \pi = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \cdots$

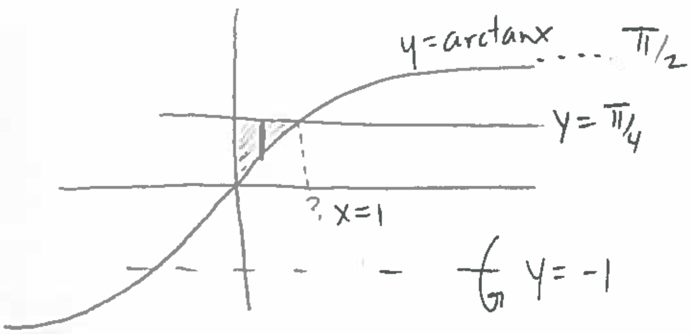
6. [18 Points] Volumes of Revolution

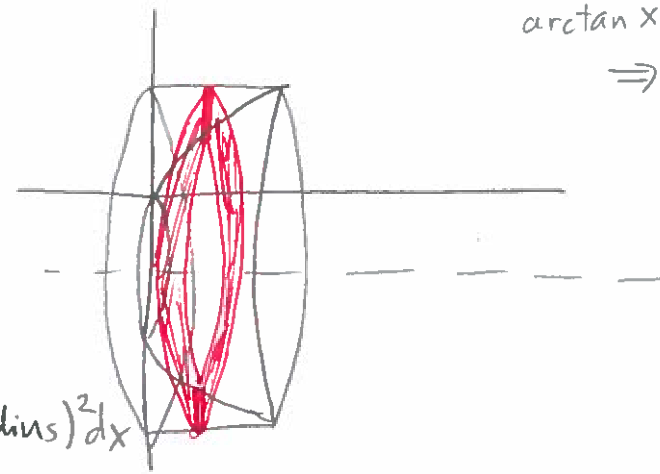
(a) Consider the region bounded by $y = \arctan x$, $y = \frac{\pi}{4}$, and $x = 0$. Rotate this region about the horizontal line $\boxed{y = -1}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Washer Method. Sketch the solid, along with one of the approximating washers.

Intersect?

$\arctan x = \frac{\pi}{4}$

$\Rightarrow x = 1$





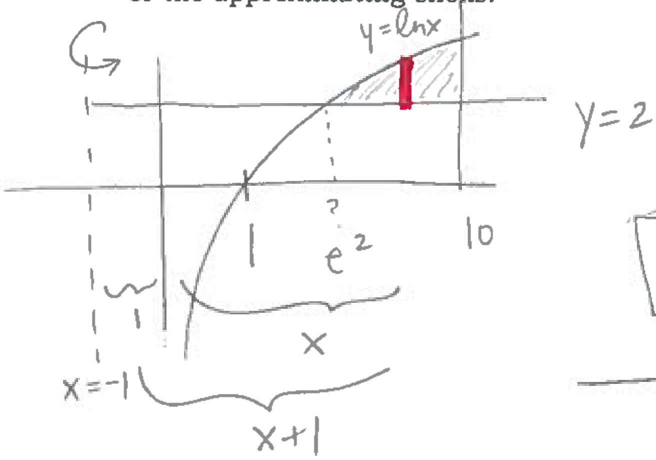
$$V = \pi \int_0^1 (\text{Outer Radius})^2 - (\text{Inner Radius})^2 \, dx$$

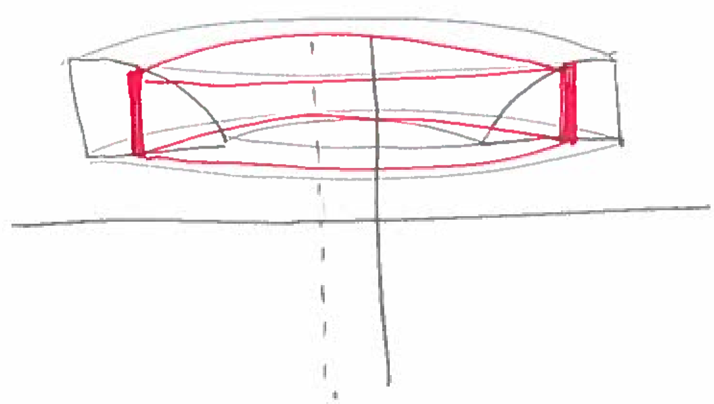
$$= \boxed{\pi \int_0^1 \left(\frac{\pi}{4}+1\right)^2 - (\arctan x + 1)^2 \, dx}$$

(b) Consider the region bounded by $y = \ln x$, $y = 2$, and $x = 10$. Rotate this region about the vertical line $\boxed{x = -1}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

Intersect? $\ln x = 2$

$\Rightarrow x = e^2$





$$V = 2\pi \int_{e^2}^{10} \text{Radius} \cdot \text{Height} \, dx$$

$$= \boxed{2\pi \int_{e^2}^{10} (x+1)(\ln x - 2) \, dx}$$

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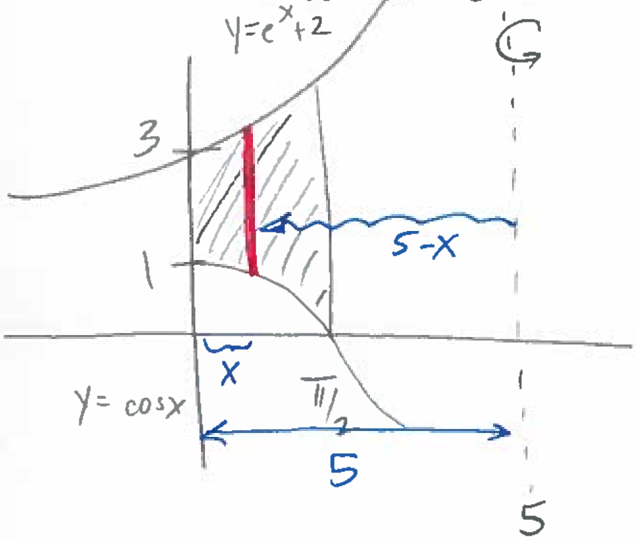
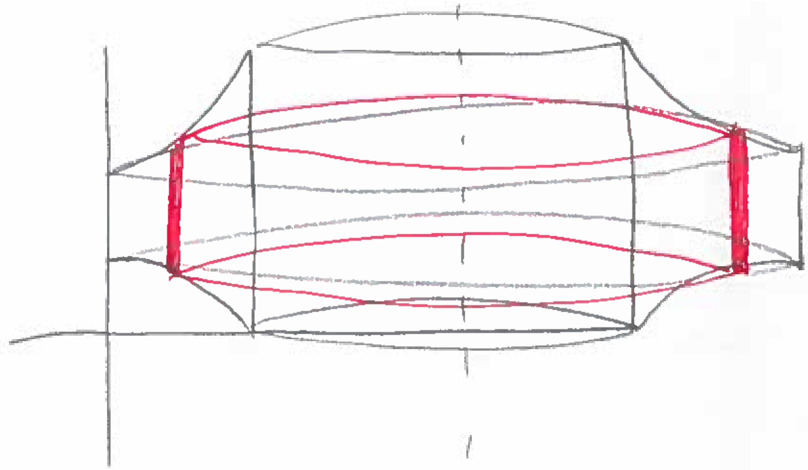
6. (Continued) Volumes of Revolution

(c) Consider the region bounded by $y = 2 + e^x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$. Rotate this region about the vertical line $\boxed{x = 5}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.





$$V = 2\pi \int_0^{\pi/2} \text{Radius} \cdot \text{Height} \, dx$$

$$= \boxed{2\pi \int_0^{\pi/2} (5-x)(e^x+2-\cos x)\, dx}$$

7. [12 Points] Parametric Equations

Consider the Parametric Curve given by $x = \ln t + \ln(1 - t^2)$ and $y = \sqrt{8}\ \arcsin t$.

COMPUTE the **Arclength** of this parametric curve for $\frac{1}{4} \le t \le \frac{1}{2}$. **SHOW** that the answer simplifies to $\boxed{\ln\left(\frac{5}{2}\right)}$

$$\frac{dx}{dt} = \frac{1}{t} - \frac{2t}{1-t^2} \qquad \frac{dy}{dt} = \frac{\sqrt{8}}{\sqrt{1-t^2}}$$

$$L = \int_{1/4}^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{1/4}^{1/2} \sqrt{\left(\frac{1}{t} - \frac{2t}{1-t^2}\right)^2 + \left(\frac{\sqrt{8}}{\sqrt{1-t^2}}\right)^2}\, dt$$

$$= \int_{1/4}^{1/2} \sqrt{\frac{1}{t^2} \underbrace{- \frac{4}{1-t^2}} + \frac{4t^2}{(1-t^2)^2} \underbrace{+ \frac{8}{1-t^2}}}\, dt = \int_{1/4}^{1/2} \sqrt{\frac{1}{t^2} + \frac{4}{1-t^2} + \frac{4t^2}{(1-t^2)^2}}\, dt$$

already $\oplus$ here

$$= \int_{1/4}^{1/2} \sqrt{\left(\frac{1}{t} + \frac{2t}{1-t^2}\right)^2}\, dt = \int_{1/4}^{1/2} \frac{1}{t} + \frac{2t}{1-t^2}\, dt = \ln|t| - \ln|1-t^2| \Big|_{1/4}^{1/2}$$

$$= \ln\left(\frac{1}{2}\right) - \ln\left(\frac{3}{4}\right) - \left[\ln\left(\frac{1}{4}\right) - \ln\left(\frac{15}{16}\right)\right] = \cancel{\ln 1}^{0} - \ln 2 - (\ln 3 - \ln 4) - \left[\cancel{\ln 1}^{0} - \ln 4 - (\ln 15 - \ln 16)\right]$$

$$= -\ln 2 - \ln 3 + \underbrace{\cancel{\ln 4 + \ln 4}}_{\cancel{2\ln 4 = \ln 16}} + \ln 15 - \cancel{\ln 16}\ \text{cancel} = \ln 15 - \underbrace{(\ln 2 + \ln 3)}_{\ln 6} = \ln 15 - \ln 6 = \ln\left(\frac{\cancel{15}^{5}}{\cancel{6}_{2}}\right) = \boxed{\ln \frac{5}{2}}\ \checkmark$$

OR easier?!

$$= \ln\left[\frac{\frac{1}{2}}{\frac{3}{4}}\right]^{4/3} - \ln\left[\frac{\frac{1}{4}}{\frac{15}{16}}\right]^{16/15} = \ln\left(\frac{2}{3}\right) - \ln\left(\frac{4}{15}\right) = \ln\left[\frac{\left(\frac{2}{3}\right)}{\frac{4}{15}}\right]^{15/4} = \boxed{\ln\left(\frac{5}{2}\right)}\ \checkmark$$

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute $\displaystyle\sum_{n=0}^{\infty} \frac{n^3(\ln 3)^n}{n!}$

See Attached Papers

OPTIONAL BONUS #2 Compute $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$

OPTIONAL BONUS #3 Find the sum of the Power Series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!}$

Bonus #1

Compute $\displaystyle\sum_{n=0}^{\infty} \frac{n^3(\ln 3)^n}{n!}$ Looks like $\displaystyle\sum_{n=0}^{\infty} \frac{n^3 x^n}{n!}$ where $x = \ln 3$

$$\sum_{n=0}^{\infty} \frac{n^3 x^n}{n!} = \sum_{n=0}^{\infty} \frac{n^3 \cdot x^{n-1} \cdot x}{n!} = x \sum_{n=0}^{\infty} \frac{n^2 \cdot n\, x^{n-1}}{n!} = x \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{n^2 x^n}{n!}\right]$$

$$= x \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{n^2 \cdot x^{n-1} \cdot x}{n!}\right] = x \frac{d}{dx}\left[x \sum_{n=0}^{\infty} \frac{n \cdot n x^{n-1}}{n!}\right]$$

$$= x \frac{d}{dx}\left[x \frac{d}{dx} \sum_{n=0}^{\infty} \frac{n x^n}{n!}\right] = x \frac{d}{dx}\left[x \frac{d}{dx} x \left(\sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!}\right)\right]$$

$$= x \frac{d}{dx}\left[x \frac{d}{dx}\, x\, \underbrace{\frac{d}{dx} \cancelto{e^x}{\sum_{n=0}^{\infty} \frac{x^n}{n!}}}_{e^x}\right] = x \frac{d}{dx}\left[x \frac{d}{dx}\left(x e^x\right)\right]$$

$$= x \frac{d}{dx}\, x\left(x e^x + e^x\right) = x \frac{d}{dx}\left(x^2 e^x + x e^x\right)$$

$$= x\left[\left(x^2 e^x + \underbrace{e^x(2x)}\right) + \underbrace{x e^x} + e^x\right] = x\left[x^2 e^x + 3x e^x + e^x\right]$$

$$= x^3 e^x + 3x^2 e^x + x e^x$$

Substitute $x = \ln 3$

$$\sum_{n=0}^{\infty} \frac{n^3(\ln 3)^n}{n!} = (\ln 3)^3 \cancelto{3}{e^{\ln 3}} + 3(\ln 3)^2 \cancelto{3}{e^{\ln 3}} + \ln 3\, \cancelto{3}{e^{\ln 3}}$$

$$= \boxed{3(\ln 3)^3 + 9(\ln 3)^2 + 3\ln 3}$$

OR $= 3\ln 3\left[(\ln 3)^2 + 3\ln 3 + 1\right] \underset{\text{OR}}{=} 3\ln 3\left[(\ln 3)^2 + \ln(27) + 1\right]$

Bonus #2

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3} \sum_{n=3}^{\infty} \frac{(-1)^n \left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$$

$$(\sqrt{3})^{2n+1}$$

$$= \sqrt{3}\left[\cancel{\arctan}\left(\frac{1}{\sqrt{3}}\right)^{\pi/6} - \left(\underbrace{\frac{1}{\sqrt{3}}}_{n=0} - \underbrace{\frac{\left(\frac{1}{\sqrt{3}}\right)^3}{3}}_{n=1} + \underbrace{\frac{\left(\frac{1}{\sqrt{3}}\right)^5}{5}}_{n=2} - \cdots\right)\right]$$

$$= \sqrt{3}\left[\pi/6 - \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}(3)} - \frac{1}{9\sqrt{3}(5)} + \cdots\right] = \sqrt{3}\left(\frac{\pi}{6}\right) - 1 + \frac{1}{9} - \frac{1}{45}$$

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \underbrace{\frac{1}{\sqrt{3}}}_{n=0} - \underbrace{\frac{\left(\frac{1}{\sqrt{3}}\right)^3}{3}}_{n=1} + \underbrace{\frac{\left(\frac{1}{\sqrt{3}}\right)^5}{5}}_{n=2} - \underbrace{\sim\sim}_{\sum_{n=3}^{\infty}}$$

$$= \sqrt{3}\left(\frac{\pi}{6}\right) - \frac{45}{45} + \frac{5}{45} - \frac{1}{45}$$

$$\boxed{= \sqrt{3}\left(\frac{\pi}{6}\right) - \frac{41}{45}}$$

Bonus #3

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} = \sum_{n=0}^{\infty} \frac{(x+2)^n}{(n+3)!} \cdot \frac{(x+2)^3}{(x+2)^3} = \frac{1}{(x+2)^3} \sum_{n=0}^{\infty} \frac{(x+2)^{n+3}}{(n+3)!}$$

$$= \frac{1}{(x+2)^3}\left[\frac{(x+2)^3}{3!} + \frac{(x+2)^4}{4!} + \frac{(x+2)^5}{5!} + \dots \right]$$

$$\boxed{= \frac{1}{(x+2)^3}\left[e^{x+2} - \left(\underbrace{1}_{n=0} + \underbrace{(x+2)}_{n=1} + \underbrace{\frac{(x+2)^2}{2!}}_{n=2} \right) \right]}$$

missing these terms

Note: $e^{x+2} = 1 + (x+2) + \frac{(x+2)^2}{2!} + \frac{(x+2)^3}{3!} + \frac{(x+2)^4}{4!} + \dots$

See Above