Name: Answer Key

Amherst College
DEPARTMENT OF MATHEMATICS
Math 121
Midterm Exam #2
March 23, 2018

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted. Do not access any webpages during this exam.

- Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{3\ln 3}$, $\sinh(\ln 3)$, or $\arctan(\sqrt{3})$ should be simplified.

- Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		40
2		10
3		8
4		18
5		24
Total		100

1. [40 Points] Compute the following integrals. Justify your work.

(a) $\int_0^{e^4} \frac{1}{x\,[16+(\ln x)^2]}\,dx = \lim_{t\to 0^+} \int_t^{e^4} \frac{1}{x\,[16+(\ln x)^2]}\,dx$

$\boxed{\begin{array}{l} u=\ln x \\ du=\frac{1}{x}dx \end{array}}$

$\boxed{\begin{array}{l} x=t \Rightarrow u=\ln t \\ x=e^4 \Rightarrow u=\ln e^4=4 \end{array}}$

$$= \lim_{t\to 0^+} \int_{\ln t}^{4} \frac{1}{16+u^2}\,du$$

$$= \lim_{t\to 0^+} \frac{1}{4}\arctan\left(\frac{u}{4}\right)\Bigg|_{\ln t}^{4}$$

$$= \lim_{t\to 0^+} \frac{1}{4}\left[\overset{\pi/4}{\arctan}\left(\overset{1}{\frac{4}{4}}\right) - \overset{(-\pi/2)}{\arctan}\left(\frac{\overset{-\infty}{\ln \overset{0^+}{t}}}{4}\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{4} \overset{--}{+} \frac{\pi}{2}\right] = \frac{1}{4}\left[\frac{3\pi}{4}\right] = \boxed{\frac{3\pi}{16}} \quad \text{Converges}$$

(b) $\int \frac{x^3+1}{x^2+1}\,dx$

Improper

$$= \int x + \overset{\text{split}}{\frac{-x+1}{x^2+1}}\,dx$$

1st Long Division

$$\begin{array}{r} x \\ x^2+1\,\overline{)\,x^3+1} \\ -(x^3+x) \\ \hline -x+1 \end{array}$$

$$= \boxed{\frac{x^2}{2} - \frac{\ln(x^2+1)}{2} + \arctan x + C}$$

1. (Continued) Compute the following integrals. Justify your work.

(c) $$\int_1^2 \frac{4}{x^2-6x+5}\,dx = \int_1^2 \frac{4}{\underbrace{(x-5)(x-1)}_{\text{proper}}}\,dx = \lim_{t\to 1^+}\int_t^2 \frac{4}{(x-5)(x-1)}\,dx$$

$$= \lim_{t\to 1^+}\int_t^2 \frac{1}{x-5} - \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1^+} \ln|x-5| - \ln|x-1| \Big|_t^2$$

$$= \lim_{t\to 1^+} \underset{\substack{\ln 3\\ \text{Finite}}}{\ln|2-5|} - \underset{\substack{\cancel{\ln 1}\\ 0}}{\ln|2-1|} \ominus \Big(\underset{\substack{\ln 4\\ \text{Finite}}}{\ln|t-5|} \ominus \underset{-\infty}{\ln|\overset{0^+}{t-1}|}\Big)$$

$$= \boxed{-\infty} \quad \text{Diverge}$$

PFD

$$\cancel{-5)}\ \cancel{(x-1)}\left[\frac{4}{\cancel{(x-5)(x-1)}} = \frac{A}{x-5} + \frac{B}{x-1}\right](x-5)(x-1)$$

$$4 = A(x-1) + B(x-5)$$
$$= (A+B)x - A - 5B$$

- $A+B=0 \Rightarrow B=-A$
- $-A-5B=4$

$$-A-5(-A)=4$$
$$4A=4$$
$$\boxed{A=1} \Rightarrow \boxed{B=-1}$$

(d) $$\int_4^\infty \frac{4}{x^2-6x+12}\,dx = \lim_{t\to\infty}\int_4^t \frac{4}{x^2-6x+12}\,dx$$

$$\boxed{\begin{array}{l} u=x-3\\ du=dx \end{array}}\ \boxed{\begin{array}{l} x=4 \Rightarrow u=1\\ x=t \Rightarrow u=t-3 \end{array}}$$

$$= \lim_{t\to\infty} 4\int_4^t \frac{1}{(x-3)^2+3}\,dx$$

$$= \lim_{t\to\infty} 4\int_1^{t-3} \frac{1}{u^2+3}\,du$$

$$= \lim_{t\to\infty} \frac{4}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Big|_1^{t-3}$$

$$= \lim_{t\to\infty} \frac{4}{\sqrt{3}}\left[\overset{\pi/2}{\overset{2}{\arctan}\Big(\overset{\infty}{\frac{t-3}{\sqrt{3}}}\Big)} - \overset{\pi/6}{\arctan\Big(\frac{1}{\sqrt{3}}\Big)}\right] = \frac{4}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{6}\right]$$

$$= \frac{4}{\sqrt{3}}\left[\frac{2\pi}{6}\right] = \frac{8\pi}{6\sqrt{3}} = \boxed{\frac{4\pi}{3\sqrt{3}}}$$

$x^2-6x+12$

$b^2-4ac = 36-4(1)(12)$
$= 36-48$
$= \ominus$

Quadratic Irreducible

$$\underbrace{(x-3)^2}_{x^2-6x+9} \underbrace{+3}_{12}$$

2. [8 Points] (a) Determine **and state** whether the following sequence **converges** or **diverges**. If it converges, compute its limit. Justify your answer. Do **not** just put down a number.

$$\left\{\left(1-\frac{4}{n^2}\right)^{n^2}\right\}_{n=1}^{\infty}$$

$$\overset{1^{\infty}}{\lim_{n\to\infty}\left(1-\frac{4}{n^2}\right)^{n^2}} = \lim_{x\to\infty}\left(1-\frac{4}{x^2}\right)^{x^2} = e^{\lim_{x\to\infty}\ln\left[\left(1-4/x^2\right)^{x^2}\right]}$$

$$= e^{\lim_{x\to\infty}\overset{\infty\cdot 0}{x^2\ln(1-4/x^2)}} \underset{\text{flip}}{=} e^{\lim_{x\to\infty}\frac{\ln(1-4/x^2)}{1/x^2}\ 0/0} \underset{\text{L'H}}{=} e^{\lim_{x\to\infty}\frac{\overset{1}{\cancel{\frac{1}{1-4/x^2}}}\left(\cancel{+}\frac{\overset{-4}{8}}{\cancel{x^3}}\right)}{\cancel{\frac{-2}{x^3}}}}$$

$$= e^{-4} = \boxed{\frac{1}{e^4}}$$ Sequences Converges

(b) Determine **and state** whether the following series **converges** or **diverges**. Justify your answer.

$$\sum_{n=1}^{\infty}\left(1-\frac{4}{n^2}\right)^{n^2}$$

The Series Diverges by nTDT because

$$\underbrace{\lim_{n\to\infty}\left(1-4/n^2\right)^{n^2}}_{\text{See Above}} = e^{-4} \neq 0$$

3. [8 Points] Find the **sum** of the following series (which does converge).

$$\sum_{n=1}^{\infty}(-1)^n \frac{7\cdot 3^{n+1}}{2^{4n-1}}$$

n=1 n=2 n=3

$$= \frac{-7\cdot 3^2}{2^3} + \frac{7\cdot 3^3}{2^7} - \frac{7\cdot 3^4}{2^{11}} + \dots$$

Geometric

$$a = \frac{-7\cdot 9}{8} = \frac{-63}{8}$$

$$r = \frac{-3}{2^4} = \frac{-3}{16}$$

$$\text{Sum} = \frac{a}{1-r} = \frac{\frac{-63}{8}}{1-\left(\frac{-3}{16}\right)} = \frac{\frac{-63}{\cancel{8}}}{\frac{19}{16}} \nearrow \frac{\overset{2}{\cancel{16}}}{19} = \boxed{\frac{-126}{19}}$$

$\frac{16}{16}$ ✓ +

4. [20 Points] Determine whether each of the following series **converges or diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\sum_{n=1}^{\infty} \sin^2\left(\frac{\pi n^4+1}{3n^4+5}\right)$

Diverges by nTDT because

$$\lim_{n\to\infty} \sin^2\left(\frac{\pi n^4+1}{3n^4+5}\right) = \sin^2\left[\lim_{n\to\infty} \frac{\pi n^4+1}{3n^4+5}\right] = \sin^2\left[\lim_{n\to\infty} \frac{\pi + \overset{\to 0}{\cancel{\frac{1}{n^4}}}}{3+\underset{0}{\cancel{\frac{5}{n^4}}}}\right]$$

$$= \left[\sin \frac{\pi}{3}\right]^2 = \left[\frac{\sqrt{3}}{2}\right]^2 = \frac{3}{4} \neq 0$$

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2(\pi n^4+1)}{3n^4+5}$ $\xrightarrow{\text{A.S.}}$ $\sum_{n=1}^{\infty} \frac{\sin^2(\pi n^4+1)}{3n^4+5}$

Bound Terms

$$\frac{\sin^2(\pi n^4+1)}{3n^4+5} \leq \frac{1}{3n^4+5} \leq \frac{1}{3n^4} \leq \frac{1}{n^4}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^4}$ Converges

p-Series $p=4>1$

$\Rightarrow$ A.S. Converges by CT

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by ACT

4. (Continued) Determine whether the following series **converges** or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(c) $\sum_{n=1}^{\infty} \frac{1}{2018} + \frac{1}{(2018)^n}$

Diverges by nTDT because

$$\lim_{n\to\infty} \frac{1}{2018} + \cancelto{0}{\frac{1}{(2018)^n}} = \frac{1}{2018} \neq 0$$

(d) $\sum_{n=1}^{\infty} \frac{1}{n^{2018}} + \frac{1}{(2018)^n} = \sum_{n=1}^{\infty} \frac{1}{n^{2018}} + \sum_{n=1}^{\infty} \frac{1}{(2018)^n}$

Converges
p-Series
$p = 2018 > 1$

Converges
Geometric Series
$|r| = \left|\frac{1}{2018}\right| = \frac{1}{2018} < 1$

O.S. Converges because the Sum of 2 Convergent Series is Convergent

5. [24 Points] Determine whether the given series is **absolutely convergent, conditionally convergent,** or **diverges.** Name any convergence test(s) you use, and justify all of your work.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{3n^5+n^2}{n^9+4}$ $\hookrightarrow \sum_{n=1}^{\infty} \frac{3n^5+n^2}{n^9+4} \approx \sum_{n=1}^{\infty} \frac{n^5}{n^9} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ Converges p-Series $p=4>1$

$$\lim_{n\to\infty} \frac{\frac{3n^5+n^2}{n^9+4}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{3n^9+n^6}{n^9+4} \frac{\left(\frac{1}{n^9}\right)}{\left(\frac{1}{n^9}\right)} = \lim_{n\to\infty} \frac{3+\cancel{\frac{1}{n^3}}^{\to 0}}{1+\cancel{\frac{4}{n^9}}^{\to 0}} = 3$$

Finite + Non-Zero

$\Rightarrow$ A.S. Converges by LCT

$\Rightarrow$ $\boxed{\text{A.C}}$ (by Definition)

5. (Continued) Determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(2n)!}{e^n\,(n^n)\,n!}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}\,[2(n+1)]!}{e^{n+1}(n+1)^{n+1}(n+1)!}}{\dfrac{\cancel{(-1)^{n}}\,(2n)!}{e^{n}(n^{n})\,n!}}\right|$$

$$(2n+2)(2n+1)\cancel{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)!}{\cancel{(2n)!}}\cdot\frac{\cancel{e^n}}{e^{n+1}}\cdot\frac{n^n}{(n+1)^{n+1}}\cdot\frac{\cancel{n!}}{(n+1)!}$$

$$\cancel{e^n}\,e \qquad (n+1)^n(n+1) \qquad (n+1)\,\cancel{n!}$$

$$= \lim_{n\to\infty}\left(\frac{2n+2}{n+1}\right)\left(\frac{2n+1}{n+1}\right)\cdot\frac{1}{e}\cdot\left(\frac{n}{n+1}\right)^{n} \to \frac{1}{e}$$

$$= \lim_{n\to\infty}\left(\frac{\overset{\nearrow 2}{2}+\overset{\nearrow 0}{2/n}}{1+\underset{\searrow 0}{1/n}}\right)\left(\frac{\overset{\nearrow 2}{2}+\overset{\nearrow 0}{1/n}}{1+\underset{\searrow 0}{1/n}}\right)\cdot\frac{1}{e^2}$$

$$= \frac{4}{e^2} < 1 \quad \text{O.S.} \ \boxed{\text{A.C.}} \ \text{by R.T.}$$

5. (Continued) Determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(c) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n-4}$ $\xrightarrow{A.S.}$ $\sum_{n=1}^{\infty}\frac{1}{5n-4} \approx \sum_{n=1}^{\infty}\frac{1}{n}$ Diverges, Harmonic p-Series $p=1$

$$\lim_{n\to\infty}\frac{\frac{1}{5n-4}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{5n-4}\frac{(1/n)}{(1/n)} = \lim_{n\to\infty}\frac{1}{5-\underset{0}{4/n}} = \frac{1}{5}$$

Finite + Non-zero

$\Rightarrow$ A.S. Diverges by LCT

AST

1) $b_n = \frac{1}{5n-4} > 0$

2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\overset{0}{1}}{\underset{\infty}{5n-4}} = 0$

3) $b_{n+1} \le b_n$

$$b_{n+1} = \frac{1}{\underbrace{5(n+1)-4}_{5n+1}} \le \frac{1}{5n-4} = b_n \checkmark$$

OR $f(x) = \frac{1}{5x-4} \Rightarrow f'(x) = \frac{\overset{\ominus}{-5}}{\underset{\oplus}{(5x-4)^2}} < 0$

Terms Decreasing

O.S. Converges by AST

$\boxed{C.C.}$

****

OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

****

OPTIONAL BONUS #1 **Compute the sum of the following series** Conjugate

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\,(n+1)+n\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\sqrt{n+1}\left(\sqrt{n+1}+\sqrt{n}\right)}\cdot\left(\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}\right)$$

$$=\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\left(\sqrt{n+1}\right)\left[\cancel{n}+1-\cancel{n}\right]} = \sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}}$$

SPLIT

$$=\sum_{n=1}^{\infty}\frac{\cancel{\sqrt{n+1}}}{\sqrt{n}\cancel{\sqrt{n+1}}}-\frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}$$

Telescoping in Disguise

$$=\left(1-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}\right)+\cdots$$

$$=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\cancel{\frac{1}{\sqrt{2}}}\right)+\left(\cancel{\frac{1}{\sqrt{2}}}-\cancel{\frac{1}{\sqrt{3}}}\right)+\left(\cancel{\frac{1}{\sqrt{3}}}-\cancel{\frac{1}{\sqrt{4}}}\right)+\ldots+\left(\cancel{\frac{1}{\sqrt{n-1}}}-\cancel{\frac{1}{\sqrt{n}}}\right)+\left(\cancel{\frac{1}{\sqrt{n}}}-\frac{1}{\sqrt{n+1}}\right)$$

STOP

$$=\lim_{n\to\infty}1-\cancel{\frac{1}{\underset{\to\infty}{\sqrt{n+1}}}}\to 0=\boxed{1}$$

"Full Sum equals limiting value of Partial Sums"

OPTIONAL BONUS #2 Compute: $\int\frac{e^{2x}}{e^{8x}-1}\,dx$

See Next Page

Bonus #2

$$\int \frac{e^{2x}}{e^{8x}-1}\,dx = \int \frac{e^{2x}}{(e^{2x})^4-1}\,dx = \frac{1}{2}\int \frac{1}{u^4-1}\,du$$

$$\boxed{\begin{aligned} u&=e^{2x}\\ du&=2e^{2x}dx\\ \tfrac{1}{2}du&=e^{2x}dx \end{aligned}}$$

$$= \frac{1}{2}\int \frac{1}{(u^2-1)(u^2+1)}\,du$$

$$= \frac{1}{2}\int \frac{1}{(u-1)(u+1)(u^2+1)}\,du \overset{\text{PFD}}{=} \frac{1}{2}\int \frac{1/4}{u-1} - \frac{\frac{1}{4}}{u+1} - \frac{\frac{1}{2}}{u^2+1}\,du$$

$$= \frac{1}{8}\ln|u-1| - \frac{1}{8}\ln|u+1| - \frac{1}{4}\arctan u + C$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| - \frac{1}{4}\arctan u + C$$

$$\boxed{= \frac{1}{2}\ln\left|\frac{e^{2x}-1}{e^{2x}+1}\right| - \frac{1}{4}\arctan(e^{2x}) + C}$$

P.F.D

$$\frac{1}{(u-1)(u+1)(u^2+1)} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{C}{u^2+1}\Bigg]\,(u-1)(u+1)(u^2+1)$$

$$1 = A(u+1)(u^2+1) + B(u-1)(u^2+1) + C\overset{(u-1)(u+1)}{(u^2-1)}$$

$$1 = A(u^3+u^2+u+1) + B(u^3-u^2+u-1) + C(u^2-1)$$

$$= \underline{Au^3} + Au^2 + \underline{Au} + A + \underline{Bu^3} - Bu^2 + \underline{Bu} - B + Cu^2 - C$$

$$= (A+B)u^3 + (A-B+C)u^2 + (A+B)u + (A-B-C)$$

- $A+B=0$ $\quad A=-B$
- $A-B+C=0$ $\quad -B-B+C=0$
- $A+B=0$ $\quad -2B+C=0$
- $A-B-C=1$ $\quad C=2B$

$$-B-B-(2B)=1$$

$$-4B=1$$

$$B=-\tfrac{1}{4} \Rightarrow A=\tfrac{1}{4}$$

$$\Rightarrow C = 2\left(-\tfrac{1}{4}\right) = -\tfrac{1}{2}$$