Answer Key

Math 121 Midterm Exam #1 March 11-14, 2021

Due Sunday, March 14, in Gradescope by 11:59 pm ET

- This is an *Open Notes* Exam. You can use course materials, homeworks problems, lecture notes, etc. that you manually worked on.
- There is **NO** *Open Internet* allowed. You can only access our Main Course Webpage.
- You are **NOT** allowed to work on or discuss these problems with other people, including the Professor or Math Fellow TA.
- Submit your final work in Gradescope in the Exam 1 entry.
- Please *show* all of your work and *justify* all of your answers. No Calculators.

Problem	Score	Possible Points
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Limits [30 Points total, 10 points per each limit] Evaluate each of the following. Please justify/simplify. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

1. $\displaystyle \lim_{x\to 1} \frac{\arctan(x-1)-\cos(\pi x)-x}{\sinh(x^2-1)-2\ln x}$ $\quad \frac{0}{0}$

$$\overset{L'H}{=} \lim_{x\to 1} \frac{\frac{1}{1+(x-1)^2} + \pi \sin(\pi x) - 1}{\cosh(x^2-1)(2x) - \frac{2}{x}}$$

$[1+(x-1)^2]^{-1}$ $\quad$ chain rule $\quad$ $\frac{2}{x} \to -2x^{-1}$

chain rule

$$\overset{L'H}{=} \lim_{x\to 1} \frac{-[1+(x-1)^2]^{-2}\cdot 2(x-1) + \pi^2 \cos(\pi x)}{\cosh(x^2-1)(2) + 2x\sinh(x^2-1)(2x) + \frac{2}{x^2}}$$

product rule

$$= \lim_{x\to 1} \frac{\frac{-2(x-1)}{[1+(x-1)^2]^2} + \pi^2 \cos(\pi x)}{2\cosh(x^2-1) + 4x^2\sinh(x^2-1) + \frac{2}{x^2}} = \boxed{-\frac{\pi^2}{4}}$$

Match!

2. $\displaystyle \lim_{x\to 0^+} x^5 \ln x$ $\quad 0\cdot(-\infty)$

$$= \lim_{x\to 0^+} \frac{\ln x}{1/x^5} \quad \frac{-\infty}{\infty} \qquad \frac{1}{x^5} \to x^{-5}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{1/x}{-5/x^6} = \lim_{x\to 0^+} \frac{1}{x}\left(-\frac{x^6}{5}\right) = \lim_{x\to 0^+} \frac{-x^5}{5} = \boxed{0}$$

Match!

3. $\displaystyle \lim_{x\to\infty} \left(1-\arcsin\left(\frac{2}{x^4}\right)\right)^{x^4}$ $\quad 1^\infty$

$$= e^{\lim_{x\to\infty} \ln\left[\left(1-\arcsin\left(\frac{2}{x^4}\right)\right)^{x^4}\right]}$$

pull down

$$= e^{\lim_{x\to\infty} x^4 \ln\left(1-\arcsin\left(2/x^4\right)\right)} \quad \infty\cdot 0$$

flip

$$= e^{\lim_{x\to\infty} \frac{\ln\left(1-\arcsin\left(2/x^4\right)\right)}{1/x^4}} \quad \frac{0}{0}$$

Show all steps

$$\overset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{1-\arcsin(2/x^4)}\cdot\left[-\frac{1}{\sqrt{1-(2/x^4)^2}}\right]\cdot\left(\frac{-8}{x^5}\right)}{-4/x^5}}$$

or $\left(-\frac{x^5}{4}\right)$

$$= \boxed{e^{-2}}$$

Match!

Integrals [70 Points total, 10 points per each integral]

4. Compute $\int_1^3 \frac{1}{\sqrt{x}\cdot\sqrt{4-x}}\,dx$ Simplify. $= \int_1^3 \frac{1}{\sqrt{x}\sqrt{4-(\sqrt{x})^2}}\,dx$

$$\boxed{\begin{gathered} u=\sqrt{x} \\ du=\frac{1}{2\sqrt{x}}\,dx \\ 2du=\frac{1}{\sqrt{x}}\,dx \end{gathered}}$$

$$\boxed{\begin{gathered} x=1 \Rightarrow u=\sqrt{1}=1 \\ x=3 \Rightarrow u=\sqrt{3} \end{gathered}}$$

$$= 2\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-u^2}}\,du$$

a-rule

$$= 2\arcsin\left(\frac{u}{2}\right)\Big|_1^{\sqrt{3}}$$

$$= 2\left[\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right)\right]$$

$\pi/3$ $\quad$ $\pi/6$

$$= 2\left[\pi/3 - \pi/6\right] = 2\left(\pi/6\right) = \boxed{\frac{\pi}{3}}$$

5. Compute $\int_e^{e^3} \frac{1}{x\,[3+(\ln x)^2]}\,dx$ Simplify.

$$\boxed{\begin{gathered} u=\ln x \\ du=\frac{1}{x}\,dx \end{gathered}}$$

$$\boxed{\begin{gathered} x=e \Rightarrow u=\ln e=1 \\ x=e^3 \Rightarrow u=\ln e^3=3 \end{gathered}}$$

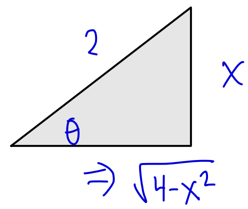
$$= \int_1^3 \frac{1}{3+u^2}\,du = \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Big|_1^3$$

$\pi/3$ $\quad$ $\sqrt{3}$ $\quad$ $\pi/6$

$$= \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{3}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{\sqrt{3}}\left[\pi/3 - \pi/6\right] = \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \boxed{\frac{\pi}{6\sqrt{3}}}$$

Integrals (Continued)



6. Show that $\int_{-2}^{2} \sqrt{4-x^2}\, dx = \boxed{2\pi}$ using Trigonometric Substitution

$$\boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\, d\theta \end{aligned}}$$

$\hookrightarrow \sin\theta = \frac{x}{2}$

$\hookrightarrow \theta = \arcsin\left(\frac{x}{2}\right)$

$$= \int_{x=-2}^{x=2} \sqrt{4-4\sin^2\theta} \cdot 2\cos\theta\, d\theta = 4\int_{x=-2}^{x=2} \cos^2\theta\, d\theta$$

$\sqrt{4(1-\sin^2\theta)}$

$\sqrt{4\cos^2\theta}$

$2\cos\theta$

$$= \cancel{4}^{\,2} \int_{x=-2}^{x=2} \frac{1+\cos(2\theta)}{\cancel{2}}\, d\theta = 2\left[\theta + \frac{\overbrace{\sin(2\theta)}^{\cancel{2}\sin\theta\cos\theta}}{\cancel{2}}\right]\Bigg|_{x=-2}^{x=2} = 2\left[\arcsin\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)\right]\Bigg|_{-2}^{2}$$

$$= 2\left[\arcsin\left(\frac{2}{2}\right) + \frac{2}{2}\cdot 0 - \left(\arcsin\left(\frac{-2}{2}\right) + \left(\frac{-2}{2}\right)\cdot 0\right)\right] = 2\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \boxed{2\pi}$$

$\frac{\pi}{2}$ $\quad 1 \quad 0 \quad -\frac{\pi}{2} \quad -1 \quad 0$

Match!

7. Compute $\int \frac{x}{\sqrt{4+x^2}}\, dx$ using Trigonometric Subsitution

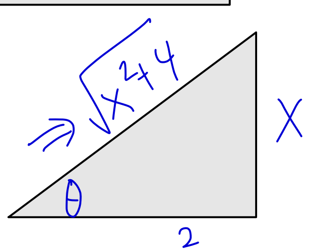
$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\, d\theta \end{aligned}}$$

$$= \int \frac{2\tan\theta}{\sqrt{4+4\tan^2\theta}} \cdot \cancel{2}\sec^{\cancel{2}}\theta\, d\theta = 2\int \tan\theta\sec\theta\, d\theta$$

$\sqrt{4(1+\tan^2\theta)}$

$\sqrt{4\sec^2\theta}$

$\cancel{2\sec\theta}$



$$= 2\int \sec\theta\tan\theta\, d\theta = 2\sec\theta + C = \cancel{2}\left(\frac{\sqrt{x^2+4}}{\cancel{2}}\right) + C = \boxed{\sqrt{x^2+4} + C}$$

Note: Makes Sense by u-sub Double Check

$$\boxed{\begin{aligned} u &= x^2+4 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \end{aligned}}$$

$$\int \frac{x}{\sqrt{4+x^2}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du = \sqrt{u} + C = \sqrt{x^2+4} + C$$ Match!

Integrals (Continued)

8. Compute $\int x^2 \arcsin x \, dx = \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx$

IBP

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = x^2\,dx \\ du = \frac{1}{\sqrt{1-x^2}}\,dx & v = \frac{x^3}{3} \end{array}}$$

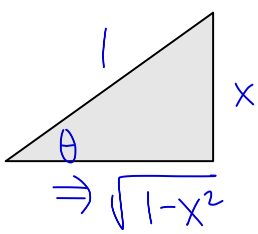
$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \frac{\sin^3\theta}{\underbrace{\sqrt{1-\sin^2\theta}}_{\sqrt{\cos^2\theta} \;=\; \cancel{\cos\theta}}}\cdot \cancel{\cos\theta}\,d\theta$$

ODD Power

Trig. Sub

$$\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{array}}$$

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \underbrace{\sin^2\theta}_{(1-\cos^2\theta)}\cdot\sin\theta\,d\theta$$



$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\int 1-u^2\,du$$

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\left(u - \frac{u^3}{3}\right) + C$$

$$\boxed{\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \\ -du = \sin\theta\,d\theta \end{array}}$$

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\left(\cos\theta - \frac{\cos^3\theta}{3}\right) + C$$

$$\boxed{= \frac{x^3}{3}\arcsin x + \frac{1}{3}\left[\sqrt{1-x^2} - \frac{\left(\sqrt{1-x^2}\right)^3}{3}\right] + C}$$

Integrals (Continued)

9. Show that $\int_1^e [\ln(x^3)]^2 \, 1 \, dx = \boxed{9e - 18}$ $= x(\ln(x^3))^2\Big|_1^e - 6\int_1^e \ln(x^3)\,dx$

IBP

$u = (\ln(x^3))^2 \quad dv = 1\,dx$

$du = 2(\ln(x^3))\frac{1}{\cancel{x^3}}(3\cancel{x^2})\,dx \quad v = \cancel{x}$

$= x(\ln(x^3))^2\Big|_1^e - 6\left[x\ln(x^3)\Big|_1^e - \int_1^e 3\,dx\right]$

$3x$

$u = \ln(x^3) \quad dv = 1\,dx$

$du = \frac{1}{x^3}(3x^2)\,dx \quad v = x$

$= x(\ln(x^3))^2\Big|_1^e - 6x\ln(x^3)\Big|_1^e + 18x\Big|_1^e$

$= e(\cancel{\ln(e^3)})^2 - \cancel{(\ln 1)}^2 - 6e\cancel{\ln(e^3)} + \cancel{6\ln 1} + 18e - 18$

$3^2 \quad 0 \quad 3 \quad 0$

$= 9e - 1\cancel{8}e + 1\cancel{8}e - 18 = \boxed{9e - 18}$ Match!

OR you can use Log Algebra to simplify first

$\int_1^e [\ln(x^3)]^2 dx = \int_1^e [3\ln x]^2 dx = 9\int_1^e (\ln x)^2 dx$

now IBP

Integrals (Continued)

u-sub

10. Compute $\int \frac{e^{2x}}{(4+e^{4x})^{\frac{3}{2}}}\,dx$ Hint: $e^{4x} = (e^{2x})^2$

$$\boxed{\begin{aligned} u &= e^{2x} \\ du &= 2e^{2x}\,dx \\ \tfrac{1}{2}\,du &= e^{2x}\,dx \end{aligned}}$$

$$= \int \frac{e^{2x}}{\left(\sqrt{4+(e^{2x})^2}\right)^3}\,dx = \frac{1}{2}\int \frac{1}{\left(\sqrt{4+u^2}\right)^3}\,du$$

Trig. Sub

$$\boxed{\begin{aligned} u &= 2\tan\theta \\ du &= 2\sec^2\theta\,d\theta \end{aligned}}$$

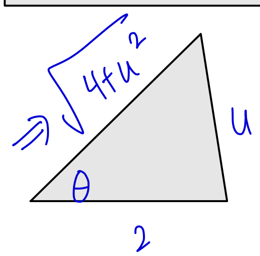
$$= \frac{1}{\cancel{2}}\int \frac{1}{\left(\sqrt{4+4\tan^2\theta}\right)^{3}} \cdot \cancel{2}\sec^2\theta\,d\theta$$

don't drop

$$\sqrt{4(1+\tan^2\theta)}$$

$$\sqrt{4\sec^2\theta}$$

$$(2\sec\theta)^3$$



$$= \frac{1}{2^3}\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{8}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{8}\int \cos\theta\,d\theta$$

$$= \frac{1}{8}\sin\theta + C = \frac{1}{8}\left[\frac{u}{\sqrt{4+u^2}}\right] + C = \boxed{\frac{1}{8}\left[\frac{e^{2x}}{\sqrt{4+e^{4x}}}\right] + C}$$