Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 121

Midterm Exam #1

February 19, 2018

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $\sinh(\ln 3)$, $e^{\ln 4}$, $\ln(e^7)$, or $e^{3\ln 3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		12
2		30
3		28
4		30
Total		100

1. [12 Points]

(a) Use implicit differentiation to **PROVE** that $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$.

Let $y = \arcsin x$

Invert $\sin y = x$

Differentiate $\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$

$$\cos y \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{+\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$ ✓

$\sin^2 y + \cos^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \pm\sqrt{1-\sin^2 y}$

$\cos y$ on $[-\pi/2, \pi/2]$ where $\arcsin x$ is defined

(b) Use implicit differentiation to **PROVE** that $\frac{d}{dx}\ln x = \frac{1}{x}$.

Let $y = \ln x$

Invert $e^y = x$

Differentiate $\frac{d}{dx}[e^y] = \frac{d}{dx}(x)$

$$e^y \cdot \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ ✓

(c) Use implicit differentiation to **PROVE** that $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$.

Let $y = \sinh^{-1} x$

Invert $\sinh y = x$

Differentiate $\frac{d}{dx}[\sinh y] = \frac{d}{dx}[x]$

$$\cosh y \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}}$ 1 $= \frac{1}{\sqrt{1+x^2}}$ ✓

$\cosh^2 y - \sinh^2 y = 1$

$\cosh^2 y = 1 + \sinh^2 y$

$\cosh y = \sqrt{1+\sinh^2 y}$

$\cosh y > 0$ for all y

2. [30 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle \lim_{x\to 0} \frac{\ln(1-x)+\sinh x}{\arctan(2x)-e^{2x}+1} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{\dfrac{-1}{1-x}+\cosh x}{\dfrac{2}{1+(2x)^2}-2e^{2x}}$ $\quad 0/0$

$2(1+4x^2)^{-1}$

$$\underset{L'H}{=} \lim_{x\to 0} \frac{\mp(1-x)^{-2}(-1)+\sinh x}{-2(1+4x^2)^{-2}(8x)-4e^{2x}} = \lim_{x\to 0} \frac{\underbrace{\dfrac{-1}{(1-x)^2}}_{\to -1}+\underbrace{\sinh x}_{\to 0}}{\underbrace{\dfrac{-16x}{(1+4x^2)^2}}_{\to 0}-\underbrace{4e^{2x}}_{\to 4}}$$

$$= \frac{-1}{-4} = \boxed{\frac{1}{4}}$$

(b) $\displaystyle \lim_{x\to\infty} (\ln x)^{\frac{6}{x}} = e^{\lim_{x\to\infty} \ln\left[(\ln x)^{6/x}\right]} = e^{\lim_{x\to\infty} \frac{6}{x}\ln(\ln x)}$

$$= e^{\lim_{x\to\infty} \frac{6\ln(\ln x)}{x}} \overset{\infty/\infty}{\underset{L'H}{=}} e^{\lim_{x\to\infty} \frac{6\cdot\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)}{1}}$$

$$= e^0 = \boxed{1}$$

2. (Continued) Evaluate the following **limit**. Please justify your answer.

(c) $\lim_{x\to\infty} \left[1 - \arctan\left(\frac{3}{x^2}\right)\right]^{x^2}$ 1^∞ $= e^{\lim_{x\to\infty} \ln\left[\left(1-\arctan\left(\frac{3}{x^2}\right)\right)^{x^2}\right]}$

$\infty \cdot 0$

$$= e^{\lim_{x\to\infty} x^2 \ln\left(1-\arctan\left(3/x^2\right)\right)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(1-\arctan\left(3/x^2\right)\right)}{1/x^2}} \quad 0/0 \quad \text{"Flip"}$$

$$\overset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{1-\arctan(3/x^2)}\cdot\left(-\frac{1}{1+(3/x^2)^2}\right)\left(\cancel{-6/x^3}\right)}{\cancel{-2/x^3}}}$$

$$= e^{\lim_{x\to\infty} \frac{1}{1-\arctan(3/x^2)}\left(-\frac{1}{1+(3/x^2)^2}\right)(3)}$$

1, (-1), 0, 0, 0

$$= e^{-3} = \boxed{\frac{1}{e^3}}$$

3. [28 Points] Compute the following **definite integral**. Please simplify your answer.

(a) Show that $\displaystyle\int_1^{e^4} \frac{\ln x}{\sqrt{x}}\,dx = 4e^2 + 4$

$$\int_1^{e^4} \ln x \cdot x^{-1/2}\,dx \overset{IBP}{=} 2\sqrt{x}\ln x\Big|_1^{e^4} - 2\int_1^{e^4} \frac{1}{\sqrt{x}}\,dx$$

$x^{-1/2}$

IBP

$$\boxed{\begin{array}{ll} u = \ln x & dv = x^{-1/2}dx \\ du = \frac{1}{x}dx & v = 2\sqrt{x} \end{array}}$$

$$= 2\sqrt{x}\ln x\Big|_1^{e^4} - 4\sqrt{x}\Big|_1^{e^4}$$

$$= 2\cancel{\sqrt{e^4}}^{\,e^2}\cdot\cancel{\ln(e^4)}^{\,4} - \cancel{2\sqrt{1}\ln 1}^{\,0} - \left[4\cancel{\sqrt{e^4}}^{\,e^2} - 4\cancel{\sqrt{1}}^{\,1}\right]$$

$$= 8e^2 - 4e^2 + 4$$

$$= \boxed{4e^2 + 4} \quad \checkmark$$

3. (Continued) Compute the following **definite integrals**. Please simplify your answer.

(b) $\int_2^{2\sqrt{3}} \frac{1}{\sqrt{16-x^2}}\,dx = \arcsin\left(\frac{x}{4}\right)\Big|_2^{2\sqrt{3}}$

$= \arcsin\left(\frac{2\sqrt{3}}{4}\right) - \arcsin\left(\frac{2}{4}\right)$

$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right)$

$= \pi/3 - \pi/6 = \boxed{\pi/6}$

(c) $\int_0^{\ln 3} \frac{e^x}{3+e^{2x}}\,dx = \int_0^{\ln 3} \frac{e^x}{3+(e^x)^2}\,dx = \int_1^3 \frac{1}{3+w^2}\,dw$

$\boxed{\begin{array}{l} w = e^x \\ dw = e^x\,dx \end{array}}$ $\boxed{\begin{array}{l} x=0 \Rightarrow w=1 \\ x=\ln 3 \Rightarrow w=e^{\ln 3}=3 \end{array}}$ $= \frac{1}{\sqrt{3}} \arctan\left(\frac{w}{\sqrt{3}}\right)\Big|_1^3$

$= \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{\cancel{3}^{\sqrt{3}}}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right]$

$= \frac{1}{\sqrt{3}}\left[\pi/3 - \pi/6\right]$

$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{6}\right] = \boxed{\frac{\pi}{6\sqrt{3}}}$

4. [30 Points] Compute the following **indefinite integral**.

(a) $\int x \arcsin x \, dx \overset{\text{IBP}}{=} \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

IBP

$u = \arcsin x \quad dv = x\,dx$

$du = \frac{1}{\sqrt{1-x^2}}dx \quad v = \frac{x^2}{2}$

Trig. Sub.

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\underbrace{\sqrt{1-\sin^2\theta}}_{\underbrace{\sqrt{\cos^2\theta}}_{\cancel{\cos\theta}}}} \cdot \cancel{\cos\theta}\, d\theta$

$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta\, d\theta$

$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2}\, d\theta$

$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{\overbrace{\sin(2\theta)}^{2\sin\theta\cos\theta}}{\cancel{2}}\right] + C$

$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\arcsin x - x\sqrt{1-x^2}\right] + C$

$= \boxed{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$

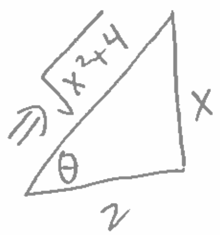
4. (Continued) Compute the following **indefinite integral.**

(b) $\displaystyle\int \frac{1}{(x^2+4)^{\frac{7}{2}}}\,dx = \int \frac{1}{\left(\sqrt{4\tan^2\theta+4}\right)^7}\cdot 2\sec^2\theta\, d\theta$

Trig. Sub $\left(\sqrt{x^2+4}\right)^7$

$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\, d\theta \end{aligned}}$$

$$= \int \frac{1}{\Big(\underbrace{\sqrt{4(\tan^2\theta+1)}}_{\sqrt{4\sec^2\theta}}\Big)^7}\cdot 2\sec^2\theta\, d\theta$$



$$= \int \frac{1}{2^7\cdot \sec^7\theta}\cdot 2\sec^2\theta\, d\theta$$

$$= \frac{2}{2^7}\int \frac{\sec^2\theta\, d\theta}{\sec^7\theta} = \frac{1}{64}\int \frac{1}{\sec^5\theta}\, d\theta = \frac{1}{64}\int \cos^5\theta\, d\theta$$

$$= \frac{1}{64}\int \underbrace{\cos^4\theta}_{(\cos^2\theta)^2}\cdot \cos\theta\, d\theta = \frac{1}{64}\int (1-\sin^2\theta)^2\cos\theta\, d\theta$$

$$\boxed{\begin{aligned} w &= \sin\theta \\ dw &= \cos\theta\, d\theta \end{aligned}}$$

$$= \frac{1}{64}\int (1-w^2)^2\, dw = \frac{1}{64}\int 1-2w^2+w^4\, dw$$

$$= \frac{1}{64}\left[w - \frac{2}{3}w^3 + \frac{w^5}{5}\right] + C = \frac{1}{64}\left[\sin\theta - \frac{2}{3}(\sin\theta)^3 + \frac{1}{5}(\sin\theta)^5\right] + C$$

$$\boxed{= \frac{1}{64}\left[\frac{x}{\sqrt{x^2+4}} - \frac{2}{3}\left(\frac{x}{\sqrt{x^2+4}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{x^2+4}}\right)^5\right] + C}$$

OR

$$= \frac{1}{64}\left[\frac{x}{\sqrt{x^2+4}} - \frac{2x^3}{3(x^2+4)^{3/2}} + \frac{x^5}{5(x^2+4)^{5/2}}\right] + C$$

4. (Continued) Compute the following **indefinite integral.** "slip-in, slip-out"

(c) $\int \ln(x^2+7)\,dx$ $= x\ln(x^2+7) - 2\int \frac{x^2 \; +7-7}{x^2+7}\,dx$

IBP

$$= x\ln(x^2+7) - 2\int \frac{\cancel{x^2+7}}{\cancel{x^2+7}}^{\,1} - \frac{7}{x^2+7}\,dx$$

$\boxed{\begin{array}{ll} u=\ln(x^2+7) & dv = 1dx \\ du = \frac{2x}{x^2+7}dx & v = x \end{array}}$

$$= x\ln(x^2+7) - 2\left[x - \cancel{7}^{\sqrt{7}} \cdot \frac{1}{\cancel{\sqrt{7}}}\arctan\left(\frac{x}{\sqrt{7}}\right)\right] + C$$

$$= \boxed{x\ln(x^2+7) - 2x + 2\sqrt{7}\arctan\left(\frac{x}{\sqrt{7}}\right) + C}$$

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute the following **indefinite integral.**

1. $\displaystyle\int \frac{1}{(x^4+4x^3+6x^2+4x+1)\sqrt{x^2+2x-8}}\,dx$

See Next Pages

OPTIONAL BONUS #2 Compute the following **indefinite integral.**

2. $\displaystyle\int \frac{\arcsin x}{x^2}\,dx$

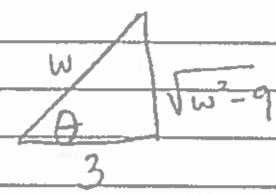
See Next Pages

Bonus #1

$$\int \frac{1}{(x^4+4x^3+6x^2+4x+1)\sqrt{x^2+2x-8}}\,dx = \int \frac{1}{(x+1)^4\sqrt{(x+1)^2-9}}\,dx$$

$$\boxed{\begin{array}{l} w = x+1 \\ dw = dx \end{array}} \quad = \int \frac{1}{w^4\sqrt{w^2-9}}\,dw = \int \frac{1}{3^4 \sec^4\theta \sqrt{\underbrace{9\sec^2\theta - 9}_{9(\sec^2\theta-1)}}}\, 3\sec\theta\cancel{\tan\theta}\, d\theta$$

$$\boxed{\begin{array}{l} w = 3\sec\theta \\ dw = 3\sec\theta\tan\theta\, d\theta \end{array}}$$



$$\underbrace{\sqrt{9\tan^2\theta}}_{\cancel{3\tan\theta}}$$

$$= \frac{1}{81}\int \frac{1}{\sec^4\theta}\sec\theta\, d\theta = \frac{1}{81}\int \frac{1}{\sec^3\theta}\, d\theta = \frac{1}{81}\int \cos^3\theta\, d\theta$$

$$= \frac{1}{81}\int \cos^2\theta\cos\theta\, d\theta = \frac{1}{81}\int (1-\sin^2\theta)\cos\theta\, d\theta = \frac{1}{81}\int (1-w^2)\, dw$$

$$\boxed{\begin{array}{l} w = \sin\theta \\ dw = \cos\theta\, d\theta \end{array}}$$

$$= \frac{1}{81}\left[w - \frac{w^3}{3}\right] + C = \frac{1}{81}\left[\sin\theta - \frac{\sin^3\theta}{3}\right] + C$$

$$\boxed{= \frac{1}{81}\left[\frac{\sqrt{w^2-9}}{w} - \frac{1}{3}\left(\frac{\sqrt{w^2-9}}{w}\right)^3\right] + C}$$

$$\text{OR} \quad \boxed{= \frac{1}{81}\left[\frac{\sqrt{(x+1)^2-9}}{x+1} - \frac{((x+1)^2-9)^{3/2}}{3(x+1)^3}\right] + C}$$

Bonus #2

$$\int \frac{\arcsin x}{x^2}\,dx = \int \arcsin x \left(x^{-2}\right)dx = \frac{-\arcsin x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\,dx$$

IBP

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = x^{-2}\,dx \\ du = \dfrac{1}{\sqrt{1-x^2}}\,dx & v = -\dfrac{1}{x} \end{array}}$$

$$= \frac{-\arcsin x}{x} + \int \frac{1}{\sin\theta \underbrace{\cancel{\sqrt{1-\sin^2\theta}}}_{\cancel{\cos\theta}}} \cdot \cancel{\cos\theta}\,d\theta$$

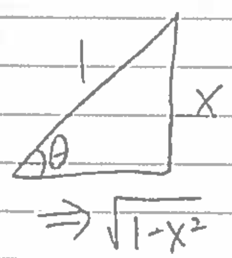
Trig Sub

$$\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{array}}$$

$$= \quad\downarrow\quad + \int \csc\theta\,d\theta$$

see (*) below

$$= \quad\downarrow\quad -\ln\left|\csc\theta + \cot\theta\right| + C$$



$$= \frac{-\arcsin x}{x} - \ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + C$$

$$= \boxed{\frac{-\arcsin x}{x} - \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C}$$

$$\csc\theta = \frac{1}{\sin\theta_{x/H}} = \frac{H}{O}$$

(*) $$\int \csc\theta\,d\theta = \int \csc\theta\left(\frac{\csc\theta + \cot\theta}{\csc\theta + \cot\theta}\right)d\theta = \int \frac{\csc^2\theta + \csc\theta\cot\theta}{\csc\theta + \cot\theta}\,d\theta$$

$w = \csc\theta + \cot\theta$

$w = -\csc\theta\cot\theta - \csc^2\theta\,d\theta$

$$= -\int \frac{1}{w}\,dw = -\ln|w| + C = -\ln\left|\csc\theta + \cot\theta\right| + C$$