

Exam #1 Review Packet Answer Key

Derivatives

1. $f(x) = \arcsin x \cdot \arctan x + \arctan(\sin(\ln x))$

$$f'(x) = \arcsin x \cdot \left[\frac{1}{1+x^2} \right] + \arctan x \cdot \left[\frac{1}{\sqrt{1-x^2}} \right] + \frac{1}{1+(\sin(\ln x))^2} \cdot \cos(\ln x) \cdot \frac{1}{x}$$

2. $f(x) = \frac{\sinh(x^2-2)}{x + \sin^{-1}x}$

$$f'(x) = \frac{(x + \sin^{-1}x) \cdot \cosh(x^2-2) (2x) - \sinh(x^2-2) \left[1 + \frac{1}{\sqrt{1-x^2}} \right]}{(x + \sin^{-1}x)^2}$$

3. $f(x) = \sin(e^{\arcsin e^x})$

$$f'(x) = \cos x (e^{\arcsin e^x}) \cdot e^{\arcsin e^x} \cdot \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x$$

4. $f(x) = \ln \left(1 - \arcsin \left(\frac{7}{x} \right) \right)$

$$f'(x) = \frac{1}{1 - \arcsin \left(\frac{7}{x} \right)} \cdot \left[\frac{-1}{\sqrt{1 - \left(\frac{7}{x} \right)^2}} \right] \cdot \left(\frac{-7}{x^2} \right)$$

5. $f(x) = \ln(1-x)$

$$f'(x) = \frac{-1}{1-x} = -(1-x)^{-1}$$

$$f''(x) = \frac{-1}{(1-x)^2} \cdot (-1) = \frac{-1}{(1-x)^2}$$

6. $f(x) = \arctan(3x)$

$$f'(x) = \frac{1}{1+(3x)^2} (3) = 3(1+9x^2)^{-1}$$

$$f''(x) = -3(1+9x^2)^{-2} (18x) = \frac{-54x}{(1+9x^2)^2}$$

7. $f(x) = \arcsin(4x)$

$$f'(x) = \frac{1}{\sqrt{1-(4x)^2}} \cdot 4 = 4(1-16x^2)^{-1/2}$$

$$f''(x) = -4 \left(\frac{1}{2}\right) (1-16x^2)^{-3/2} (-32x) = \frac{64x}{(1-16x^2)^{3/2}}$$

Proofs

8. Let $y = \arcsin x$

Invert $\sin y = x$

Differentiate $\frac{d}{dx} [\sin y] = \frac{d}{dx} [x]$

$$\cos y \cdot \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-(\sin y)^2}} = \frac{1}{\sqrt{1-x^2}} \quad \square$

\oplus on $-\pi/2 \leq y \leq \pi/2$

9. Let $y = \arctan x$

Invert $\tan y = x$

Differentiate $\frac{d}{dx} [\tan y] = \frac{d}{dx} [x]$

$$\sec^2 y \cdot \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{(1 + \tan^2)^2} = \frac{1}{1 + x^2} \quad \square$

10. Let $y = \sinh^{-1} x$

Invert $\sinh y = x$

Differentiate $\frac{d}{dx} [\sinh y] = \frac{d}{dx} [x]$

$$\cosh y \cdot \frac{dy}{dx} = 1$$

Identity

$$\cosh^2 y - \sinh^2 y = 1$$

$$\cosh y = \sqrt{1 + \sinh^2 y}$$

Solve $\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + (\sinh)^2}} = \frac{1}{\sqrt{1 + x^2}} \quad \square$

11. Let $y = \arcsin(5x)$

Invert $\sin y = 5x$

Differentiate $\frac{d}{dx} [\sin y] = \frac{d}{dx} [5x]$

$$\cos y \cdot \frac{dy}{dx} = 5$$

Solve $\frac{dy}{dx} = \frac{5}{\cos y} = \frac{5}{\sqrt{1 - \sin^2 y}} = \frac{5}{\sqrt{1 - (\sin y)^2}} = \frac{5}{\sqrt{1 - (5x)^2}} = \frac{5}{\sqrt{1 - 25x^2}} \quad \square$

12. Let $y = \ln x$

Invert $e^y = x$

Differentiate $\frac{d}{dx} [e^y] = \frac{d}{dx} [x]$

$$e^y \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x} \quad \square$

13. Let $y = \arctan(3x)$

Invert $\tan y = 3x$

Differentiate $\frac{d}{dx} [\tan y] = \frac{d}{dx} [3x]$

$$\sec^2 y \cdot \frac{dy}{dx} = 3$$

Solve $\frac{dy}{dx} = \frac{3}{\sec^2 y} = \frac{3}{1 + \tan^2 y} = \frac{3}{1 + (3x)^2} = \frac{3}{1 + 9x^2}$ \square

14. $\int \frac{1}{9+x^2} dx = \int \frac{1}{9 \left[1 + \frac{x^2}{9}\right]} dx = \frac{1}{9} \int \frac{1}{1 + \frac{x^2}{9}} dx$ *create perfect square*

$$= \frac{1}{9} \int \frac{1}{1 + \left(\frac{x}{3}\right)^2} dx = \frac{3}{9} \int \frac{1}{1+u^2} du = \frac{1}{3} \arctan u + C$$

$$= \frac{1}{3} \arctan\left(\frac{x}{3}\right) + C \quad \square$$

$$u = \frac{x}{3}$$

$$du = \frac{1}{3} dx$$

$$3du = dx$$

15. $\int \frac{1}{\sqrt{16-x^2}} dx = \int \frac{1}{\sqrt{16 \left(1 - \frac{x^2}{16}\right)}} dx = \int \frac{1}{\sqrt{16} \sqrt{1 - \frac{x^2}{16}}} dx$ *create perfect square*

$$= \frac{1}{4} \int \frac{1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}} dx = \int \frac{1}{\sqrt{1-u^2}} du = \arcsin u + C$$

$$= \arcsin\left(\frac{x}{4}\right) + C \quad \square$$

$$u = \frac{x}{4}$$

$$du = \frac{1}{4} dx$$

Limits

$$\begin{aligned}
 16. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x &= e^{\lim_{x \rightarrow \infty} \ln \left[\left(1 + \frac{1}{x}\right)^x \right]} = e^{\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right)} = e^{\lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}}} \\
 &\stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{1}{x}} \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}} = e^1 = \boxed{e}
 \end{aligned}$$

$$\begin{aligned}
 17. \lim_{x \rightarrow 0} \frac{7xe^x - \arctan(7x)}{\sinh x + \ln(1-x)} &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{7xe^x + 7e^x - \frac{1}{1+(7x)^2}}{\cosh x + \frac{1}{1-x}(-1)} \\
 &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{7xe^x + 7e^x + 7e^x + 7(1+49x^2)^{-2}(98x)}{\sinh x + (1-x)(-1)} \\
 &\stackrel{\text{simplify}}{=} \lim_{x \rightarrow 0} \frac{7xe^x + 14e^x + \frac{686x}{(1+49x^2)^2}}{\sinh x - \frac{1}{(1-x)^2}} = \frac{14}{-1} = \boxed{-14}
 \end{aligned}$$

$$\begin{aligned}
 18. \lim_{x \rightarrow 0} \frac{\cosh(4x) - 1 - \arctan(4x) + 4x}{\ln(1-x) + \arcsin x} &\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{4\sinh(4x) - \frac{4}{1+(4x)^2} + 4}{-\frac{1}{1-x} + \frac{1}{\sqrt{1-x^2}}} \\
 &\quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
 &\quad \quad \quad -(1-x)^{-1} \quad \quad (1-x^2)^{-1/2}
 \end{aligned}$$

$$= \lim_{x \rightarrow 0} \frac{16 \cosh(4x) + 4(1+16x^2)^{-2} (32x)}{(1-x)^{-2} (-1) + \frac{1}{2} (1-x^2)^{-3/2} (-2x)}$$

simplify

$$= \lim_{x \rightarrow 0} \frac{16 \cosh(4x) + \frac{128x}{(1+16x^2)^2}}{(1-x)^{-2} + \frac{x}{(1-x^2)^{3/2}}} = \frac{16}{-1} = \boxed{-16}$$

19. $\lim_{x \rightarrow 0} \frac{1 - e^{-4x} - \arctan(4x)}{x^2} \frac{0}{0} = \lim_{x \rightarrow 0} \frac{+4e^{-4x} - \frac{4}{1+(4x)^2}}{2x} \frac{0}{0}$

$$= \lim_{x \rightarrow 0} \frac{-16e^{-4x} + 4(1+16x^2)^{-2} (32x)}{2} = \lim_{x \rightarrow 0} \frac{-16e^{-4x} + \frac{128x}{(1+16x^2)^2}}{2} = \frac{-16}{2} = \boxed{-8}$$

20. $\lim_{x \rightarrow 0^+} x^3 \cdot \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-3}} \frac{-\infty}{\infty} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-3}{x^4}} = \lim_{x \rightarrow 0^+} \frac{-x^3}{3} = \boxed{0}$

21. $\lim_{x \rightarrow 0^+} x \ln\left(\frac{1}{x}\right) = \lim_{x \rightarrow 0^+} \frac{\ln\left(\frac{1}{x}\right)}{\frac{1}{x}} \frac{\infty}{\infty} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0^+} \frac{1}{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) = \lim_{x \rightarrow 0^+} x = \boxed{0}$

22. $\lim_{x \rightarrow \infty} x^{\frac{1}{x^2}} = e^{\lim_{x \rightarrow \infty} \ln\left[x^{\frac{1}{x^2}}\right]} = e^{\lim_{x \rightarrow \infty} \frac{1}{x^2} \ln x} = e^{\lim_{x \rightarrow \infty} \frac{\ln x}{x^2} \frac{\infty}{\infty}}$

$$\stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{2x}{1}} \cdot \frac{1}{2x}} = e^{\lim_{x \rightarrow \infty} \frac{1}{2x^2}} = e^0 = \boxed{1}$$

$$23. \lim_{x \rightarrow 0^+} (1-2x)^{1/x} = e^{\lim_{x \rightarrow 0^+} \ln[(1-2x)^{1/x}]} = e^{\lim_{x \rightarrow 0^+} \frac{1}{x} \ln(1-2x)}$$

$$= e^{\lim_{x \rightarrow 0^+} \frac{\ln(1-2x)}{x}} \stackrel{\%}{=} e^{\lim_{x \rightarrow 0^+} \frac{1}{1-2x} \cdot (-2)} = e^{-2}$$

$$24. \lim_{x \rightarrow \infty} (x^3+1)^{1/\ln x} = e^{\lim_{x \rightarrow \infty} \ln[(x^3+1)^{1/\ln x}]} = e^{\lim_{x \rightarrow \infty} \frac{1}{\ln x} \cdot \ln(x^3+1)}$$

$$= e^{\lim_{x \rightarrow \infty} \frac{\ln(x^3+1)}{\ln x}} \stackrel{\%}{=} e^{\lim_{x \rightarrow \infty} \frac{1}{x^3+1} (3x^2) \cdot \frac{x}{1}} = e^{\lim_{x \rightarrow \infty} \frac{3x^3}{x^3+1}}$$

$$\stackrel{\%}{=} e^{\lim_{x \rightarrow \infty} \frac{9x^2}{3x^2}} = e^3$$

$$25. \lim_{x \rightarrow \infty} x^2 \sin\left(\frac{1}{x^2}\right) \stackrel{\infty \cdot 0}{=} \lim_{x \rightarrow \infty} \frac{\sin(1/x^2)}{1/x^2} \stackrel{\%}{=} \lim_{x \rightarrow \infty} \frac{\cos(1/x^2) \cdot (-2/x^3)}{-2/x^3} = 1$$

Flip

$$26. \lim_{x \rightarrow 0^+} \sqrt{x} \ln x \stackrel{0 \cdot (-\infty)}{=} \lim_{x \rightarrow 0^+} \frac{\ln x}{1/\sqrt{x}} \stackrel{\%}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-1/2x^{3/2}} = \lim_{x \rightarrow 0^+} -2x^{3/2} = 0$$

Flip

$$27. \lim_{x \rightarrow \infty} \left(1 - \frac{3}{x}\right)^{4x} = e^{\lim_{x \rightarrow \infty} \ln\left[\left(1 - \frac{3}{x}\right)^{4x}\right]} = e^{\lim_{x \rightarrow \infty} 4x \ln\left(1 - \frac{3}{x}\right)}$$

$$= e^{\lim_{x \rightarrow \infty} \frac{4 \ln\left(1 - \frac{3}{x}\right)}{1/x}} = e^{\lim_{x \rightarrow \infty} \frac{4 \cdot \frac{1}{1-3/x} \cdot \left(-\frac{3}{x^2}\right)}{-1/x^2}} = e^{-12}$$

easier to leave 4 here.

Flip

$$\begin{aligned}
 28. \lim_{x \rightarrow \infty} \left(1 + \ln\left(1 - \frac{6}{x^2}\right)\right)^{x^2} &= e^{\lim_{x \rightarrow \infty} \ln \left[\left(1 + \ln\left(1 - \frac{6}{x^2}\right)\right)^{x^2} \right]} \\
 &= e^{\lim_{x \rightarrow \infty} x^2 \ln\left(1 + \ln\left(1 - \frac{6}{x^2}\right)\right)} = e^{\lim_{x \rightarrow \infty} \frac{\ln\left(1 + \ln\left(1 - \frac{6}{x^2}\right)\right)}{\frac{1}{x^2}}} \\
 &\stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \frac{1}{1 + \ln\left(1 - \frac{6}{x^2}\right)} \cdot \frac{1}{-\frac{6}{x^2} \cdot \left(\frac{12}{x^3}\right)}} = e^{\frac{12}{-2}} = e^{-6}
 \end{aligned}$$

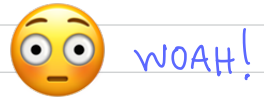
$$\begin{aligned}
 29. \lim_{x \rightarrow \infty} \left(1 - \arctan\left(\frac{5}{x^4}\right)\right)^{3x^4} &= e^{\lim_{x \rightarrow \infty} \ln \left[\left(1 - \arctan\left(\frac{5}{x^4}\right)\right)^{3x^4} \right]} \\
 &= e^{\lim_{x \rightarrow \infty} 3x^4 \ln\left(1 - \arctan\left(\frac{5}{x^4}\right)\right)} = e^{\lim_{x \rightarrow \infty} \frac{3 \ln\left(1 - \arctan\left(\frac{5}{x^4}\right)\right)}{\frac{1}{x^4}}} \\
 &\stackrel{\text{L'H}}{=} e^{\lim_{x \rightarrow \infty} \frac{3}{1 - \arctan\left(\frac{5}{x^4}\right)} \cdot \frac{(-1)}{1 + \left(\frac{5}{x^4}\right)^2} \cdot \frac{(-20)}{x^5}} = e^{-15}
 \end{aligned}$$

CHALLENGE!

$$\begin{aligned}
 30. \lim_{x \rightarrow \infty} \left(\sqrt{1 - \frac{5}{x^2}} - \sinh\left(\frac{1}{x^2}\right)\right)^{5x^2} &= e^{\lim_{x \rightarrow \infty} \ln \left[\left(\sqrt{1 - \frac{5}{x^2}} - \sinh\left(\frac{1}{x^2}\right)\right)^{5x^2} \right]} \\
 &= e^{\lim_{x \rightarrow \infty} 5x^2 \ln\left(\sqrt{1 - \frac{5}{x^2}} - \sinh\left(\frac{1}{x^2}\right)\right)} = e^{\lim_{x \rightarrow \infty} \frac{5 \ln\left(\sqrt{1 - \frac{5}{x^2}} - \sinh\left(\frac{1}{x^2}\right)\right)}{\frac{1}{x^2}}}
 \end{aligned}$$

$$= \lim_{x \rightarrow \infty} e^{\frac{5 \left(\frac{1}{\sqrt{1-5/x^2} - \sinh(1/x^2)} \right) \cdot \left[\frac{1/2}{2\sqrt{1-5/x^2}} \left(\frac{10}{x^3} \right) - \cosh(1/x^2) \left(\frac{-2}{x^3} \right) \right]}{-2/x^3}}$$

$$= e^{5 \left(\frac{-5}{2} - 1 \right)} = e^{-\frac{35}{2}}$$



$$31. \lim_{x \rightarrow \infty} \left(e^{1/x^8} - \frac{8}{x^8} \right)^{x^8} = e^{\lim_{x \rightarrow \infty} \ln \left[\left(e^{1/x^8} - \frac{8}{x^8} \right)^{x^8} \right]}$$

$$= e^{\lim_{x \rightarrow \infty} x^8 \cdot \ln \left(e^{1/x^8} - \frac{8}{x^8} \right)} = e^{\lim_{x \rightarrow \infty} \frac{\ln \left(e^{1/x^8} - \frac{8}{x^8} \right)}{1/x^8}}$$

$$= e^{\lim_{x \rightarrow \infty} \frac{1}{e^{1/x^8} - \frac{8}{x^8}} \left[e^{1/x^8} \left(\frac{-8}{x^9} \right) + \frac{64}{x^9} \right]}$$



$$= e^{\lim_{x \rightarrow \infty} \frac{1}{e^{1/x^8} - \frac{8}{x^8}} \left[e^{1/x^8} - 8 \right]} = e^{1-8} = e^{-7}$$

$$32. \lim_{x \rightarrow \infty} \left(\frac{x}{x+3} \right)^x = e^{\lim_{x \rightarrow \infty} \ln \left[\left(\frac{x}{x+3} \right)^x \right]} = e^{\lim_{x \rightarrow \infty} x \ln \left(\frac{x}{x+3} \right)}$$

$$= e^{\lim_{x \rightarrow \infty} \frac{\ln \left(\frac{x}{x+3} \right)}{1/x}} = e^{\lim_{x \rightarrow \infty} \frac{1}{\left(\frac{x}{x+3} \right)} \left[\frac{(x+3)(1) - x(1)}{(x+3)^2} \right]} = e^{\lim_{x \rightarrow \infty} \frac{1}{\left(\frac{x}{x+3} \right)} \left[\frac{(x+3) - x}{(x+3)^2} \right]} = e^{\lim_{x \rightarrow \infty} \frac{1}{\left(\frac{x}{x+3} \right)} \left[\frac{3}{(x+3)^2} \right]}$$

$$= e^{\lim_{x \rightarrow \infty} \frac{x+3}{x} \left(\frac{3}{(x+3)^2} \right)^{-x^2}} = e^{\lim_{x \rightarrow \infty} \frac{-3x^{-\infty}}{x+3}}$$

$$\stackrel{L'H}{=} e^{\lim_{x \rightarrow \infty} \frac{-3}{1}} = e^{-3}$$

Integrals

$$33. \int (e^x + x)^2 dx = \int e^{2x} + 2xe^x + x^2 dx = \frac{e^{2x}}{2} + 2 \int xe^x dx + \frac{x^3}{3} + C$$

IBP

$$\begin{array}{l} u = x \quad dv = e^x dx \\ du = dx \quad v = e^x \end{array}$$

$$= \frac{e^{2x}}{2} + 2 \left[xe^x - \int e^x dx \right] + \frac{x^3}{3} + C$$

$$= \frac{e^{2x}}{2} + 2xe^x - 2e^x + \frac{x^3}{3} + C$$

$$34. \int x \sin^2 x dx = x \left(\frac{x}{2} - \frac{\sin(2x)}{4} \right) - \int \left(\frac{x}{2} - \frac{\sin(2x)}{4} \right) dx$$

IBP

$$\begin{array}{l} u = x \quad dv = \sin^2 x dx \\ du = dx \quad v = \int \sin^2 x dx \leftarrow \text{See Below} \\ \quad = \frac{x}{2} - \frac{\sin(2x)}{4} \end{array}$$

$$= \frac{x^2}{2} - \frac{x \sin(2x)}{4} - \frac{x^2}{4} + \frac{1}{4} \int \sin(2x) dx \quad \leftarrow \text{u-sub?!}$$

$$= \frac{x^2}{2} - \frac{x \sin(4x)}{4} - \frac{x^2}{4} - \frac{1}{8} \cos(2x) + C$$

$$= \frac{x^2}{4} - \frac{x \sin(4x)}{4} - \frac{1}{8} \cos(2x) + C$$

$$\begin{aligned} (*) \int \sin^2 x dx &= \int \frac{1 - \cos(2x)}{2} dx \quad \text{Half-Angle.} \\ &= \frac{1}{2} \int (1 - \cos(2x)) dx \\ &= \frac{1}{2} \left[x - \frac{\sin(2x)}{2} \right] + C \end{aligned}$$

Note: you could instead substitute for $\sin^2 x$ right away using Half-Angle and then IBP. Either is fine.

$$35. \int \frac{1}{\sqrt{25-x^2}} dx = \arcsin\left(\frac{x}{5}\right) + C$$

Quick a-rule

OR, prove the a-rule with Algebra + u-sub.

$$36. \int \frac{1}{x^2 + 25} dx = \frac{1}{5} \arctan\left(\frac{x}{5}\right) + C$$

Quick a-rule

or, prove the a-rule with Algebra + u-sub

$$37. \int \frac{1}{x \sqrt{9 - (\ln x)^2}} dx = \int \frac{1}{\sqrt{9 - w^2}} dw = \arcsin\left(\frac{w}{3}\right) + C = \arcsin\left(\frac{\ln x}{3}\right) + C$$

$$w = \ln x \\ dw = \frac{1}{x} dx$$

$$38. \int x \arcsin x dx \stackrel{IBP}{=} \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx$$

IBP

$$u = \arcsin x \quad dv = x dx \\ du = \frac{1}{\sqrt{1-x^2}} dx \quad v = \frac{x^2}{2}$$

$$\frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx$$

Trig. Sub

Don't Forget

$$= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cdot \cos \theta d\theta$$

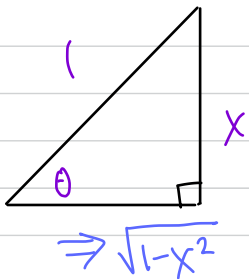
cancel

$$\sqrt{\cos^2 \theta} \\ \cos \theta$$

Trig. Sub.

$$x = \sin \theta \\ dx = \cos \theta d\theta$$

$$\theta = \arcsin x$$



$$\Rightarrow \cos \theta = \frac{A}{H} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

$$= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \sin^2 \theta d\theta$$

Now Half-Angle Identity

EVEN Power

$$= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{1 - \cos(2\theta)}{2} d\theta$$

$$= \frac{x^2}{2} \arcsin x - \frac{1}{4} \int (1 - \cos(2\theta)) d\theta$$

$$= \frac{x^2}{2} \arcsin x - \frac{1}{4} \left[\theta - \frac{\sin(2\theta)}{2} \right] + C$$

Now Double Angle Identity

$$= \frac{x^2}{2} \arcsin x - \frac{1}{4} \theta + \frac{1}{4} \sin \theta \cos \theta + C$$

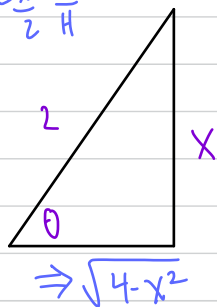
$$= \frac{x^2}{2} \arcsin x - \frac{\arcsin x}{4} + \frac{x \sqrt{1-x^2}}{4} + C$$

$$39. \int \frac{1}{(4-x^2)^{3/2}} dx = \int \frac{1}{(\sqrt{4-x^2})^3} dx \stackrel{\text{don't drop}}{=} \int \frac{1}{(\sqrt{4-4\sin^2\theta})^3} \cdot 2\cos\theta d\theta$$

Trig. Sub.

$$\begin{aligned} X &= 2\sin\theta \\ dx &= 2\cos\theta d\theta \end{aligned}$$

$$\sin\theta = \frac{X}{2} = \frac{O}{H}$$



$$\Rightarrow \tan\theta = \frac{X}{\sqrt{4-x^2}}$$

$$= \int \frac{1}{(\sqrt{4(1-\sin^2\theta)})^3} \cdot 2\cos\theta d\theta = \int \frac{1}{(\sqrt{4\cos^2\theta})^3} \cdot 2\cos\theta d\theta$$

$$= \int \frac{1}{2^3 \cos^3\theta} \cdot 2\cos\theta d\theta = \frac{2}{8} \int \frac{\cos\theta}{\cos^2\theta} d\theta = \frac{1}{4} \int \frac{1}{\cos^2\theta} d\theta$$

$$= \frac{1}{4} \int \sec^2\theta d\theta = \frac{1}{4} \tan\theta + C = \frac{1}{4} \left[\frac{X}{\sqrt{4-x^2}} \right] + C$$

$$40. \int_1^e \ln x dx = \int_1^e \ln x \cdot 1 dx = x \ln x \Big|_1^e - \int_1^e x \cdot \frac{1}{x} dx = x \ln x \Big|_1^e - x \Big|_1^e$$

IBP

$$\begin{aligned} u &= \ln x & dv &= 1 dx \\ du &= \frac{1}{x} dx & v &= x \end{aligned}$$

$$= e \ln e - \ln 1 - (e - 1) = e - e + 1 = 1$$

$$41. \int \frac{\ln(2x^5)}{x^2} dx = \int \ln(2x^5) x^{-2} dx = -\frac{\ln(2x^5)}{x} + 5 \int \frac{1}{x^2} dx$$

IBP

$$u = \ln(2x^5) \quad dv = x^{-2}$$

$$\begin{aligned} du &= \frac{1}{2x^5} (10x^4) & v &= \frac{x^{-1}}{-1} \\ &= \frac{5}{x} & &= -\frac{1}{x} \end{aligned}$$

$$= -\frac{\ln(2x^5)}{x} + 5 \left(\frac{x^{-1}}{-1} \right) + C$$

$$= -\frac{\ln(2x^5)}{x} - \frac{5}{x} + C$$

$$42. \int \cos^5 x dx = \int \cos^4 x \cdot \cos x dx = \int (\cos^2 x)^2 \cdot \cos x dx = \int (1 - \sin^2 x)^2 \cdot \cos x dx$$

ODD
ISOLATE
CONVERT
Finish with u-sub

$$\begin{aligned} u &= \sin x \\ du &= \cos x dx \end{aligned}$$

$$= \int (1-u^2)^2 du = \int 1 - 2u^2 + u^4 du = u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \sin x - \frac{2}{3} \sin^3 x + \frac{\sin^5 x}{5} + C$$

$$43. \int \ln^2(x^{20}) dx = \int (\ln(x^{20}))^2 \cdot \overset{1}{1} dx = x [\ln(x^{20})]^2 - 40 \int \ln(x^{20}) dx$$

IBP

$$u = (\ln(x^{20}))^2 \quad dv = 1 dx$$

$$du = 2(\ln(x^{20})) \cdot \frac{1}{x^{20}} \cdot 20x^{19} dx \quad v = x$$

cancel

$$= x [\ln(x^{20})]^2 - 40 \left[x \ln(x^{20}) - \int 20 dx \right]$$

$$= x [\ln(x^{20})]^2 - 40x \ln(x^{20}) + 800x + C$$

IBP

$$u = \ln(x^{20}) \quad dv = 1 dx$$

$$du = \frac{1}{x^{20}} (20x^{19}) \quad v = x$$

Note: OR you could use Log Algebra to Simplify First, then do a Double IBP.

$$\text{OR} \int (\ln(x^{20}))^2 dx = \int [20 \ln x]^2 dx = 400 \int (\ln x)^2 dx = \dots$$

$$44. \int \sin^5 \cos^2 x dx = \int \sin^4 x \cdot \cos^2 x \cdot \overset{\text{ISOLATE}}{\sin x} dx = \int (\sin^2 x)^2 \cdot \cos^2 x \cdot \overset{\text{CONVERT}}{\sin x} dx$$

$$= \int (1 - \cos^2 x)^2 \cdot \cos^2 x \cdot \sin x dx = - \int (1 - u^2)^2 \cdot u^2 du = - \int (1 - 2u^2 + u^4) u^2 du$$

FINISH U-SUB

$$= - \int u^2 - 2u^4 + u^6 du = - \left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right] + C$$

$$= - \frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$$

NOTE: Don't Mix Half-Angle Identity in ODD Power Case

$$45. \int \sin^2 x \cos^3 x dx = \int \sin^2 x \cdot \cos^2 x \cdot \overset{\text{ISOLATE}}{\cos x} dx = \int \sin^2 x (1 - \sin^2 x) \cdot \overset{\text{CONVERT}}{\cos x} dx$$

$$u = \sin x$$

$$du = \cos x dx$$

$$= \int u^2 (1 - u^2) du = \int u^2 - u^4 du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

FINISH U-SUB

$$46. \int_{e^{\sqrt{3}}}^{e^3} \frac{1}{x(9+(\ln x)^2)} dx = \int \frac{1}{9+u^2} du = \frac{1}{3} \arctan\left(\frac{u}{3}\right) \Big|_{\sqrt{3}}^3$$

$$\boxed{u = \ln x}$$

$$\boxed{du = \frac{1}{x} dx}$$

$$\boxed{x = e^{\sqrt{3}} \Rightarrow u = \ln e^{\sqrt{3}} = \sqrt{3}}$$

$$\boxed{x = e^3 \Rightarrow u = \ln e^3 = 3}$$

$$= \frac{1}{3} \left[\arctan\left(\frac{3}{3}\right) - \arctan\left(\frac{\sqrt{3}}{3}\right) \right]$$

$\swarrow \pi/4$ $\swarrow \pi/6$

$$= \frac{1}{3} \left[\frac{\pi}{4} - \frac{\pi}{6} \right] = \frac{1}{3} \left[\frac{3\pi}{12} - \frac{2\pi}{12} \right] = \frac{1}{3} \left(\frac{\pi}{12} \right) = \boxed{\frac{\pi}{36}}$$

$$47. \int_1^e \frac{1}{x(1+(\ln x)^2)^{3/2}} dx = \int_{x=1}^{x=e} \frac{1}{(1+u^2)^{3/2}} du = \int_{x=1}^{x=e} \frac{1}{(\sqrt{1+u^2})^3} du = \int_{x=1}^{x=e} \frac{1}{(\sqrt{1+\tan^2\theta})^3} \cdot \sec^2\theta d\theta$$

or change limits?

$$\boxed{u = \ln x}$$

$$\boxed{du = \frac{1}{x} dx}$$

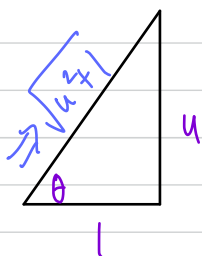
$$= \int_{x=1}^{x=e} \frac{\sec^2\theta}{\sec^3\theta} d\theta = \int_{x=1}^{x=e} \frac{1}{\sec\theta} d\theta = \int_{x=1}^{x=e} \cos\theta d\theta = \sin\theta \Big|_{x=1}^{x=e}$$

Trig Sub

$$\boxed{u = \tan\theta}$$

$$\boxed{du = \sec^2\theta d\theta}$$

$$= \frac{u}{\sqrt{u^2+1}} \Big|_{x=1}^{x=e} = \frac{\ln x}{\sqrt{1+(\ln x)^2}} \Big|_1^e = \frac{\ln e}{\sqrt{1+(\ln e)^2}} - \frac{\ln 1}{\sqrt{1+(\ln 1)^2}}$$



$$= \frac{1}{\sqrt{1+1}} = \boxed{\frac{1}{\sqrt{2}}} = \boxed{\frac{\sqrt{2}}{2}}$$

$$48. \int_0^{\pi/2} \frac{\cos x}{(1+\sin^2 x)^{7/2}} dx = \int_0^1 \frac{1}{(\sqrt{1+w^2})^7} dw = \int_{w=0}^{w=1} \frac{1}{(\sqrt{1+\tan^2\theta})^7} \cdot \sec^2\theta d\theta$$

don't drop

$$\boxed{w = \sin x}$$

$$\boxed{dw = \cos x dx}$$

$$\boxed{x=0 \Rightarrow w = \sin 0 = 0}$$

$$\boxed{x=\pi/2 \Rightarrow w = \sin \pi/2 = 1}$$

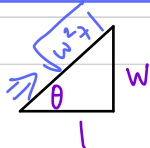
$$= \int_{w=0}^{w=1} \frac{\sec^2\theta}{\sec^7\theta} d\theta = \int_{w=0}^{w=1} \frac{1}{\sec^5\theta} d\theta = \int_{w=0}^{w=1} \cos^5\theta d\theta$$

Trig Sub

$$\boxed{w = \tan\theta}$$

$$\boxed{dw = \sec^2\theta d\theta}$$

$$= \int_{w=0}^{w=1} \cos^4\theta \cdot \cos\theta d\theta = \int_{w=0}^{w=1} (\cos^2\theta)^2 \cos\theta d\theta = \int_{w=0}^{w=1} (1-\sin^2\theta)^2 \cos\theta d\theta$$



another u-sub

$$\begin{aligned} u &= \sin \theta \\ du &= \cos \theta d\theta \end{aligned}$$

$$= \int_{w=0}^{w=1} (1-u^2)^2 du = \int_{w=0}^{w=1} (1-2u^2+u^4) du = u - \frac{2u^3}{3} + \frac{u^5}{5} \Big|_{w=0}^{w=1}$$

$$= \sin \theta - \frac{2}{3} \sin^3 \theta + \frac{\sin^5 \theta}{5} \Big|_{w=0}^{w=1} = \frac{w}{\sqrt{w^2+1}} - \frac{2}{3} \left[\frac{w}{\sqrt{w^2+1}} \right]^3 + \frac{1}{5} \left[\frac{w}{\sqrt{w^2+1}} \right]^5 \Big|_0^1$$

$$= \frac{1}{\sqrt{2}} - \frac{2}{3} \left(\frac{1}{\sqrt{2}} \right)^3 + \frac{1}{5} \left(\frac{1}{\sqrt{2}} \right)^5 = \frac{1}{\sqrt{2}} - \frac{2}{3} \left(\frac{1}{2\sqrt{2}} \right) + \frac{1}{5} \left(\frac{1}{4\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}} = \frac{60}{60\sqrt{2}} - \frac{20}{60\sqrt{2}} + \frac{3}{60\sqrt{2}} = \frac{43}{60\sqrt{2}} \quad \text{wow!}$$

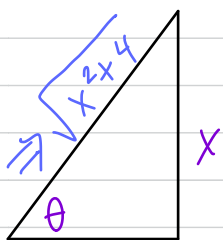
$$49. \int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} dx = \arcsin\left(\frac{x}{2}\right) \Big|_1^{\sqrt{3}} = \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$50. \int \frac{1}{(x^2+4)^2} dx = \int \frac{1}{(4\tan^2\theta+4)^2} \cdot 2\sec^2\theta d\theta = \int \frac{1}{(4\sec^2\theta)^2} \cdot 2\sec^2\theta d\theta = \frac{2}{16} \int \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

Trig. Sub.

$$\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta d\theta \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{x}{2} \\ \Rightarrow \theta &= \arctan\left(\frac{x}{2}\right) \end{aligned}$$



$$= \frac{1}{8} \int \frac{1}{\sec^2\theta} d\theta = \frac{1}{8} \int \cos^2\theta d\theta = \frac{1}{8} \int \frac{1+\cos(2\theta)}{2} d\theta$$

$$= \frac{1}{16} \int (1+\cos(2\theta)) d\theta = \frac{1}{16} \left[\theta + \frac{\sin(2\theta)}{2} \right] + C = \frac{1}{16} \left[\theta + \sin\theta \cdot \cos\theta \right] + C$$

$$= \frac{1}{16} \left[\arctan\left(\frac{x}{2}\right) + \frac{x}{\sqrt{x^2+4}} \cdot \frac{2}{\sqrt{x^2+4}} \right] + C$$

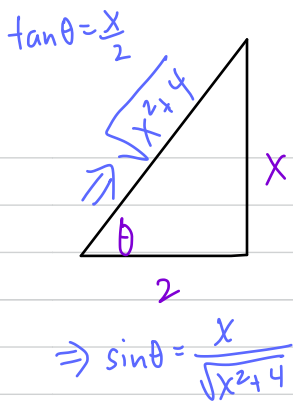
$$\begin{aligned} \Rightarrow \sin\theta &= \frac{x}{\sqrt{x^2+4}} \\ \Rightarrow \cos\theta &= \frac{2}{\sqrt{x^2+4}} \end{aligned}$$

$$= \frac{1}{16} \left[\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4} \right] + C$$

$$51. \int \frac{1}{(x^2+4)^{7/2}} dx = \int \frac{1}{(\sqrt{x^2+4})^7} dx = \int \frac{1}{(4\tan^2\theta+4)^{7/2}} \cdot 2\sec^2\theta d\theta$$

Trig. Sub

$$\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta d\theta \end{aligned}$$



$$= \int \frac{1}{(\sqrt{4\sec^2\theta})^7} \cdot 2\sec^2\theta d\theta = \frac{2}{2^7} \int \frac{\sec^2\theta}{\sec^7\theta} d\theta = \frac{1}{2^6} \int \frac{1}{\sec^5\theta} d\theta$$

$$= \frac{1}{64} \int \cos^5\theta d\theta = \frac{1}{64} \int \cos^4\theta \cdot \cos\theta d\theta = \frac{1}{64} \int (\cos^2\theta)^2 \cos\theta d\theta$$

ODD.
ISOLATE
CONVERT

$$= \frac{1}{64} \int (1-\sin^2\theta)^2 \cos\theta d\theta = \frac{1}{64} \int (1-w^2)^2 dw = \frac{1}{64} \int 1-2w^2+w^4 dw$$

FINISH U-SUB

$$= \frac{1}{64} \left[w - \frac{2w^3}{3} + \frac{w^5}{5} \right] + C = \frac{1}{64} \left[\sin\theta - \frac{2\sin^3\theta}{3} + \frac{\sin^5\theta}{5} \right] + C$$

$$= \frac{1}{64} \left[\frac{x}{\sqrt{x^2+4}} - \frac{2}{3} \left(\frac{x}{\sqrt{x^2+4} \right)^3 + \frac{1}{5} \left(\frac{x}{\sqrt{x^2+4} \right)^5 \right] + C$$

\uparrow or $\frac{x^3}{(x^2+4)^{3/2}}$ or $\frac{x^5}{(x^2+4)^{5/2}}$

52. $\int_{-1}^0 x^4 \arcsin x dx = \frac{x^5}{5} \arcsin x \Big|_{-1}^0 - \frac{1}{5} \int_{-1}^0 \frac{x^5}{\sqrt{1-x^2}} dx$ don't drop.

IBP

$$u = \arcsin x \quad dv = x^4 dx$$

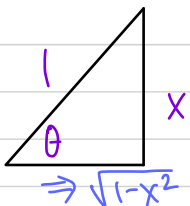
$$du = \frac{1}{\sqrt{1-x^2}} dx \quad v = \frac{x^5}{5}$$

$$= \frac{x^5}{5} \arcsin x \Big|_{-1}^0 - \frac{1}{5} \int_{x=-1}^{x=0} \frac{\sin^5\theta}{\sqrt{1-\sin^2\theta}} \cos\theta d\theta$$

Trig. Sub.

$$x = \sin\theta$$

$$dx = \cos\theta d\theta$$



$$= \frac{x^5}{5} \arcsin x \Big|_{-1}^0 - \frac{1}{5} \int_{x=-1}^{x=0} \sin^5\theta d\theta$$

ODD power

$$= \frac{x^5}{5} \arcsin x \Big|_{-1}^0 - \frac{1}{5} \int_{x=-1}^{x=0} \sin^4\theta \cdot \sin\theta d\theta$$

$\left(\sin^2\theta \right)^2$

$$= \frac{x^5}{5} \arcsin x \Big|_{-1}^0 - \frac{1}{5} \int_{x=-1}^{x=0} (1-\cos^2\theta)^2 \cdot \sin\theta d\theta$$

$$= \frac{x^5}{5} \arcsin x \Big|_{-1}^0 + \frac{1}{5} \int_{x=-1}^{x=0} (1-w^2)^2 dw$$

$1-2w^2+w^4$

$$w = \cos\theta$$

$$dw = -\sin\theta d\theta$$

$$-dw = \sin\theta d\theta$$

$$= \frac{x^5}{5} \arcsin x \Big|_{-1}^0 + \frac{1}{5} \left[w - \frac{2w^3}{3} + \frac{w^5}{5} \right] \Big|_{x=-1}^{x=0}$$

$$= \frac{x^5}{5} \arcsin x \Big|_{-1}^0 + \frac{1}{5} \left[\cos \theta - \frac{2}{3} \cos^3 \theta + \frac{\cos^5 \theta}{5} \right] \Big|_{x=-1}^{x=0}$$

$$= \frac{x^5}{5} \arcsin x \Big|_{-1}^0 + \frac{1}{5} \left[\sqrt{1-x^2} - \frac{2}{3} (\sqrt{1-x^2})^3 + \frac{1}{5} (\sqrt{1-x^2})^5 \right] \Big|_{-1}^0$$

$$= \frac{1}{5} \left[0 - \frac{(-1)^5}{5} \arcsin(-1) \right] + \frac{1}{5} \left[\left(\sqrt{1} - \frac{2}{3} (\sqrt{1})^3 + \frac{1}{5} (\sqrt{1})^5 \right) - (0 - 0 + 0) \right]$$

$$= \frac{1}{5} \left(-\frac{\pi}{2} \right) + \frac{1}{5} \left(1 - \frac{2}{3} + \frac{1}{5} \right) = -\frac{\pi}{10} + \frac{1}{5} \left(\frac{15}{15} - \frac{10}{15} + \frac{3}{15} \right) = \frac{8}{75} - \frac{\pi}{10}$$

$$53 \int \frac{\sinh x}{\sqrt{16 - \cosh^2 x}} dx = \int \frac{1}{\sqrt{16 - w^2}} dw = \arcsin \left(\frac{w}{4} \right) + C = \arcsin \left(\frac{\cosh x}{4} \right) + C$$

$$w = \cosh x \\ dw = \sinh x dx$$

$$54. \int_0^1 x \tan^{-1}(x^2) dx = \frac{1}{2} \int_0^1 \tan^{-1} w \cdot 1 dw = \frac{1}{2} \left[w \tan^{-1} w \Big|_0^1 - \int_0^1 \frac{w}{1+w^2} dw \right]$$

$$w = x^2 \\ dw = 2x dx \\ \frac{1}{2} dw = x dx$$

$$x=0 \Rightarrow w=0 \\ x=1 \Rightarrow w=1$$

same by chance.

$$= \frac{1}{2} \left[w \tan^{-1} w \Big|_0^1 - \frac{1}{2} \int_1^2 \frac{1}{p} dp \right]$$

$$= \frac{1}{2} \left[w \tan^{-1} w \Big|_0^1 - \frac{1}{2} \ln |p| \Big|_1^2 \right]$$

$$= \frac{1}{2} \left[\tan^{-1} 1 - \tan^{-1} 0 - \frac{1}{2} (\ln 2 - \ln 1) \right]$$

$$= \frac{\pi}{8} - \frac{\ln 2}{4}$$

$$u = \tan^{-1} w \quad dv = 1 dw \\ dw = \frac{1}{1+w^2} dw \quad v = w$$

$$p = 1+w^2 \\ dp = 2w dw \\ \frac{1}{2} dp = w dw$$

$$w=0 \Rightarrow p=1 \\ w=1 \Rightarrow p=2$$

Note: Or just go straight for IBP right away $u = \arctan(x^2) \quad dv = x dx$

$$55. \int \frac{x^2}{x^6+1} dx = \int \frac{x^2}{(x^3)^2+1} dx = \frac{1}{3} \int \frac{1}{w^2+1} dw = \frac{1}{3} \arctan w + C$$

$$\begin{aligned} w &= x^3 \\ dw &= 3x^2 dx \\ \frac{1}{3} dw &= x^2 dx \end{aligned}$$

$$= \frac{1}{3} \arctan(x^3) + C$$

$$56. \int_1^{e^2} x \ln \sqrt{x} dx = \frac{x^2}{2} \ln \sqrt{x} \Big|_1^{e^2} - \frac{1}{4} \int_1^{e^2} \frac{x^2}{x} dx = \frac{x^2}{2} \ln \sqrt{x} \Big|_1^{e^2} - \frac{x^2}{8} \Big|_1^{e^2}$$

IBP

$$u = \ln \sqrt{x} \quad dv = x dx$$

$$\begin{aligned} du &= \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} dx \quad v = \frac{x^2}{2} \\ &= \frac{1}{2x} dx \end{aligned}$$

$$= \frac{e^4}{2} \ln \sqrt{e^2} - \frac{1}{2} \ln \sqrt{1} - \frac{1}{8} (e^4 - 1)$$

$$= \frac{e^4}{2} - \frac{e^4}{8} + \frac{1}{8} = \frac{3e^4}{8} + \frac{1}{8} = \frac{3e^4+1}{8}$$

Note: you can also simplify first using log Algebra, then use IBP.

$$\text{OR} \int_1^{e^2} x \ln \sqrt{x} dx = \int_1^{e^2} x \ln [x^{1/2}] dx = \frac{1}{2} \int_1^{e^2} x \ln x dx = \frac{1}{2} \left[\frac{x^2}{2} \ln x \Big|_1^{e^2} - \frac{1}{2} \int_1^{e^2} x dx \right]$$

IBP

$$u = \ln x \quad dv = x dx$$

$$du = \frac{1}{x} dx \quad v = \frac{x^2}{2}$$

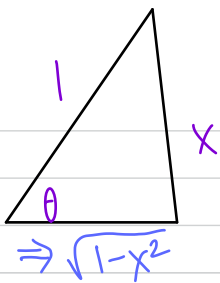
$$= \frac{1}{2} \left[\frac{x^2}{2} \ln x \Big|_1^{e^2} - \frac{x^2}{4} \Big|_1^{e^2} \right] = \frac{x^2}{4} \ln x \Big|_1^{e^2} - \frac{x^2}{8} \Big|_1^{e^2}$$

$$= \frac{e^4}{4} (\ln e^2 - \ln 1) - \frac{1}{8} (e^4 - \frac{1}{8}) = \frac{e^4}{2} - \frac{e^4}{8} + \frac{1}{8} = \frac{3e^4+1}{8} \text{ Match}$$

$$57. \int \frac{x^2}{(1-x^2)^{3/2}} dx = \int \frac{x^2}{(\sqrt{1-x^2})^3} dx = \int \frac{\sin^2 \theta}{(\sqrt{1-\sin^2 \theta})^3} \cdot \cos \theta d\theta = \int \frac{\sin^2 \theta}{\cos^3 \theta} \cos \theta d\theta$$

$$x = \sin \theta$$

$$dx = \cos \theta d\theta \Rightarrow \theta = \arcsin x$$



$$= \int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta = \int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + C$$

Identity

$$= \frac{x}{\sqrt{1-x^2}} - \arcsin x + C$$

$$\Rightarrow \tan \theta = \frac{x}{\sqrt{1-x^2}}$$

$$58. \int_1^e (\ln x)^2 dx = x(\ln x)^2 \Big|_1^e - 2 \int_1^e \ln x dx = x(\ln x)^2 \Big|_1^e - 2 \left[x \ln x \Big|_1^e - \int_1^e 1 dx \right]$$

warning: $(\ln x)^2 \neq 2 \ln x$
No!

$$= x(\ln x)^2 \Big|_1^e - 2x \ln x \Big|_1^e + 2x \Big|_1^e \quad \text{or pull all limits off...}$$

IBP

$$\begin{aligned} u &= (\ln x)^2 & dv &= 1 dx \\ du &= 2 \ln x \cdot \frac{1}{x} dx & v &= x \end{aligned}$$

$$= e(\ln e)^2 - (\ln 1)^2 - 2(e \ln e - \ln 1) + 2(e - 1)$$

$$= e - 2e + 2e - 2 = e - 2$$

IBP

$$\begin{aligned} u &= \ln x & dv &= 1 dx \\ du &= \frac{1}{x} dx & v &= x \end{aligned}$$

Note: you can also split $\int_1^e \ln x \cdot \ln x dx$ but still double IBP.

$$59. \int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} + \frac{1}{x^2+9} dx = \arcsin\left(\frac{x}{2}\right) + \frac{1}{3} \arctan\left(\frac{x}{3}\right) \Big|_0^{\sqrt{3}} \quad \text{a-rules}$$

$$= \arcsin \frac{\sqrt{3}}{2} - \arcsin 0 + \frac{1}{3} \left[\arctan \left(\frac{\sqrt{3}}{3} \right) - \arctan 0 \right]$$

$$= \frac{\pi}{3} + \frac{\pi}{18} = \frac{6\pi}{18} + \frac{\pi}{18} = \frac{7\pi}{18}$$

$$60. \int \frac{x^2}{\sqrt{16-x^2}} dx = \int \frac{16 \sin^2 \theta}{\sqrt{16-16 \sin^2 \theta}} \cdot 4 \cos \theta d\theta = 16 \int \sin^2 \theta d\theta = 16 \int \frac{1 - \cos(2\theta)}{2} d\theta$$

EVEN

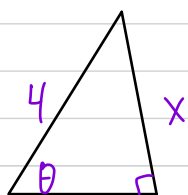
Half-Angle

Trig-Sub.

$$\begin{aligned} x &= 4 \sin \theta \\ dx &= 4 \cos \theta d\theta \end{aligned}$$

$$\begin{aligned} \sqrt{16-16 \sin^2 \theta} &= \sqrt{16 \cos^2 \theta} \\ &= 4 \cos \theta \end{aligned}$$

$$\sin \theta = \frac{x}{4} \Rightarrow \theta = \arcsin\left(\frac{x}{4}\right)$$



$$\Rightarrow \sqrt{16-x^2}$$

$$\Rightarrow \cos \theta = \frac{\sqrt{16-x^2}}{4}$$

$$= 8 \int 1 - \cos(2\theta) d\theta = 8 \left[\theta - \frac{\sin(2\theta)}{2} \right] + C$$

$$= 8 \left[\arcsin\left(\frac{x}{4}\right) - \left(\frac{x}{4}\right) \left(\frac{\sqrt{16-x^2}}{4}\right) \right] + C$$

$$\text{or} = 8 \arcsin\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C$$

$$61. \int x^3 \sqrt{9-x^2} dx = \int 27 \sin^3 \theta \cdot \sqrt{9-9\sin^2 \theta} \cdot 3 \cos \theta d\theta = 243 \int \sin^3 \theta \cos^2 \theta d\theta$$

ODD Power.

$$\begin{aligned} x &= 3 \sin \theta \\ dx &= 3 \cos \theta d\theta \end{aligned}$$

$$\begin{aligned} &\sqrt{9(1-\sin^2 \theta)} \\ &\sqrt{9 \cos^2 \theta} \\ &3 \cos \theta \end{aligned}$$

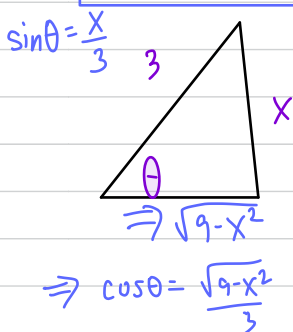
or 3⁵

$$= 243 \int \sin^2 \theta \cos^2 \theta \cdot \sin \theta d\theta = 243 \int (1-\cos^2 \theta) \cdot \cos^2 \theta \cdot \sin \theta d\theta$$

$$= -243 \int (1-w^2) w^2 dw = -243 \int w^2 - w^4 dw = -243 \left[\frac{w^3}{3} - \frac{w^5}{5} \right] + C$$

$$= -243 \left[\frac{\cos^3 \theta}{3} - \frac{\cos^5 \theta}{5} \right] + C = -243 \left[\frac{1}{3} \left(\frac{\sqrt{9-x^2}}{3}\right)^3 - \frac{1}{5} \left(\frac{\sqrt{9-x^2}}{3}\right)^5 \right] + C$$

$$= -3 (9-x^2)^{3/2} + \frac{1}{5} (9-x^2)^{5/2} + C$$



$$\begin{aligned} w &= \cos \theta \\ dw &= -\sin \theta d\theta \\ -dw &= \sin \theta d\theta \end{aligned}$$

Note: OR Inverted Substitution also works
OR IBP works

$$62. \int \frac{x^2}{x^2+3} dx = \int \frac{x^2+3-3}{x^2+3} dx = \int \frac{x^2+3}{x^2+3} - \frac{3}{x^2+3} dx$$

$$\text{slip-in/slip-out split/split} = x - \frac{3}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

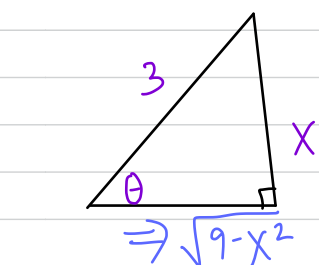
$$= x - \sqrt{3} \arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

Note: Trig. Sub also works

63. $\int_{-3}^3 \sqrt{9-x^2} dx = \int_{x=-3}^{x=3} \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta d\theta = 9 \int_{x=-3}^{x=3} \cos^2\theta d\theta$ Half-Angle

Triq. Sub.
 $x = 3\sin\theta$
 $dx = 3\cos\theta d\theta$

$\sin\theta = \frac{x}{3} \Rightarrow \theta = \arcsin\left(\frac{x}{3}\right)$



$\Rightarrow \cos\theta = \frac{\sqrt{9-x^2}}{3}$

$= 9 \int_{x=-3}^{x=3} \frac{1+\cos(2\theta)}{2} d\theta = \frac{9}{2} \int_{x=-3}^{x=3} 1+\cos(2\theta) d\theta$

$= \frac{9}{2} \left[\theta + \frac{\sin(2\theta)}{2} \right] \Big|_{x=-3}^{x=3} = \frac{9}{2} \left[\arcsin\left(\frac{x}{3}\right) + \left(\frac{x}{3}\right) \frac{\sqrt{9-x^2}}{3} \right] \Big|_{-3}^3$

$= \frac{9}{2} \left[\arcsin\left(\frac{3}{3}\right) + \frac{3\sqrt{9-9}}{9} - \left(\arcsin\left(\frac{-3}{3}\right) - \left(\frac{-3}{3}\right) \frac{\sqrt{9-9}}{3} \right) \right]$

$= \frac{9}{2} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = \frac{9\pi}{2}$

Makes Sense: Area of half circle of Radius 3, $x^2+y^2=9$
 $\pi r^2 \rightarrow 9\pi$ Half $\rightarrow \frac{9\pi}{2}$

64. $\int_1^e \sqrt{x} \ln x dx = \frac{2}{3} x^{3/2} \ln x \Big|_1^e - \frac{2}{3} \int_1^e \frac{x^{3/2}}{x} dx = \frac{2}{3} x^{3/2} \ln x \Big|_1^e - \frac{4}{9} x^{3/2} \Big|_1^e$

IBP
 $u = \ln x \quad dv = x^{1/2} dx$
 $du = \frac{1}{x} dx \quad v = \frac{2}{3} x^{3/2}$

$= \frac{2}{3} \left[e^{3/2} \cdot \ln e - \ln 1 \right] - \frac{4}{9} \left[e^{3/2} - 1 \right]$
 $= \frac{2}{3} e^{3/2} - \frac{4}{9} e^{3/2} + \frac{4}{9} = \frac{2}{9} e^{3/2} + \frac{4}{9}$

65. $\int \frac{x+3}{\sqrt{4-x^2}} dx = \int \frac{x}{\sqrt{4-x^2}} + 3 \int \frac{1}{\sqrt{4-x^2}} dx$

$u = 4-x^2$
 $du = -2x dx$
 $-\frac{1}{2} du = x dx$

$= -\frac{1}{2} \int \frac{1}{\sqrt{u}} du + 3 \arcsin\left(\frac{x}{2}\right) + C$

$= -\frac{1}{2} \left(\frac{u^{-1/2}}{-1/2} \right) + 3 \arcsin\left(\frac{x}{2}\right) + C$

$$= -\sqrt{4-x^2} + 3\arcsin\left(\frac{x}{2}\right) + C$$

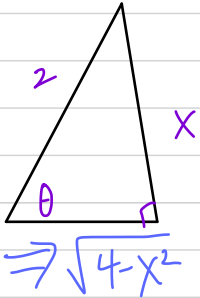
OR Trig. Sub.

Trig. Sub

$$\int \frac{x+3}{\sqrt{4-x^2}} dx = \int \frac{2\sin\theta + 3}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta d\theta = \int 2\sin\theta + 3 d\theta = -2\cos\theta + 3\theta + C$$

$$\begin{aligned} X &= 2\sin\theta \\ dx &= 2\cos\theta d\theta \end{aligned}$$

$$\sin\theta = \frac{x}{2} \Rightarrow \theta = \arcsin\left(\frac{x}{2}\right)$$



$$\begin{aligned} &\sqrt{4(1-\sin^2\theta)} \\ &\sqrt{4\cos^2\theta} \\ &2\cos\theta \end{aligned}$$

$$= -2\left(\frac{\sqrt{4-x^2}}{2}\right) + 3\arcsin\left(\frac{x}{2}\right) + C$$

$$= -\sqrt{4-x^2} + 3\arcsin\left(\frac{x}{2}\right) + C \quad \text{Match!}$$

CHALLENGE

$$67. \int (\arcsin x)^2 dx = x(\arcsin x)^2 - 2 \int \frac{x \arcsin x}{\sqrt{1-x^2}} dx$$

IBP

$$\begin{aligned} u &= (\arcsin x)^2 & dv &= 1 dx \\ du &= 2\arcsin x \cdot \frac{1}{\sqrt{1-x^2}} dx & v &= x \end{aligned}$$

$$\begin{aligned} w &= \arcsin x & \Rightarrow x &= \sin w \\ dw &= \frac{1}{\sqrt{1-x^2}} dx \end{aligned}$$

$$= x(\arcsin x)^2 - 2 \int (\sin^T w) \cdot w dw \quad \text{NEW IBP}$$

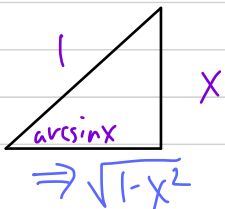
$$= x(\arcsin x)^2 - 2 \left[-w \cos w + \int \cos w dw \right]$$

$$= x(\arcsin x)^2 + 2w \cos w - 2\sin w + C$$

$$= x(\arcsin x)^2 + 2(\arcsin x) \cos(\arcsin x) - 2\sin(\arcsin x) + C$$

$$\begin{aligned} u &= w & dw &= \sin w dw \\ du &= dw & w &= -\cos w \end{aligned}$$

$$= x(\arcsin x)^2 + 2\arcsin x \sqrt{1-x^2} - 2x + C$$



$$\Rightarrow \cos(\arcsin x) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$