Math 121 Final Exam Spring 2020

1(a). $\int_0^1 x^2 \arcsin x \, dx = \frac{x^3}{3}\arcsin x \Big|_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\, dx$

IBP

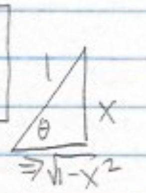
$$\boxed{\begin{array}{ll} u = \arcsin x & dv = x^2 dx \\ du = \frac{1}{\sqrt{1-x^2}} & v = \frac{x^3}{3} \end{array}}$$

$$= \quad \downarrow \quad -\frac{1}{3}\int_{x=0}^{x=1} \frac{\sin^3\theta}{\underbrace{\sqrt{1-\sin^2\theta}}_{\sqrt{\cos^2\theta} \;=\; \cancel{\cos\theta}}} \cancel{\cos\theta}\, d\theta$$

$$= \quad -\frac{1}{3}\int_{x=0}^{x=1} \sin^3\theta \, d\theta$$

Trig. Sub

$$\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta \, d\theta \end{array}}$$



$$= \quad \downarrow \quad -\frac{1}{3}\int_{x=0}^{x=1} \underbrace{\sin^2\theta}_{(1-\cos^2\theta)} \sin\theta \, d\theta$$

$$\boxed{\begin{array}{l} w = \cos\theta \\ dw = -\sin\theta \, d\theta \\ -dw = \sin\theta \, d\theta \end{array}}$$

$$= \quad \downarrow \quad \overset{--}{+}\frac{1}{3}\int_{x=0}^{x=1} 1 - w^2 \, dw$$

$$= \quad \downarrow \quad +\frac{1}{3}\left(w - \frac{w^3}{3}\right)\Big|_{x=0}^{x=1}$$

$$= \frac{x^3}{3}\arcsin x \Big|_0^1 + \frac{\cos\theta}{3} - \frac{\cos^3\theta}{9}\Big|_{x=0}^{x=1}$$

$$= \frac{x^3}{3}\arcsin x \Big|_0^1 + \frac{\sqrt{1-x^2}}{3} - \frac{\left(\sqrt{1-x^2}\right)^3}{9}\Big|_0^1$$

$$= \frac{1}{3}\left[\overset{\pi/2}{\arcsin(1)} - \overset{0}{\arcsin 0}\right] + \left(0 - 0 - \left(\frac{\sqrt{1}}{3} - \frac{(\sqrt{1})^3}{9}\right)\right)$$

$$-\left(\frac{1}{3} - \frac{1}{9}\right)$$

$$-\frac{2}{9}$$

$$= \boxed{\frac{\pi}{6} - \frac{2}{9}}$$ Match ✓

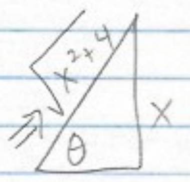
1(b) $\displaystyle\int \frac{1}{(x^2+4)^2}\,dx = \int \frac{1}{\left(\sqrt{x^2+4}\right)^2}\,dx = \int \frac{1}{\left(\underbrace{\sqrt{4\tan^2\theta+4}}_{\underbrace{4(\tan^2\theta+1)}_{4\sec^2\theta}}\right)^4}\,2\sec^2\theta\,d\theta$

$\boxed{\begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta\,d\theta \end{array}}$

$\tan\theta = \frac{x}{2}$

$\theta = \arctan\left(\frac{x}{2}\right)$

$\displaystyle = \int \frac{1}{\left(\underbrace{\sqrt{4\sec^2\theta}}_{(2\sec\theta)^4}\right)^4}\,2\sec^2\theta\,d\theta$



$\displaystyle = \frac{1}{8}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \frac{1}{8}\int \frac{1}{\sec^2\theta}\,d\theta = \frac{1}{8}\int \cos^2\theta\,d\theta$

$\displaystyle = \frac{1}{8}\int \frac{1+\cos(2\theta)}{2}\,d\theta = \frac{1}{16}\int 1+\cos(2\theta)\,d\theta = \frac{1}{16}\left[\theta + \frac{\overset{2\sin\theta\cos\theta}{\sin(2\theta)}}{\cancel{2}}\right] + C$

$\displaystyle = \frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \left(\frac{x}{\sqrt{x^2+4}}\right)\left(\frac{2}{\sqrt{x^2+4}}\right)\right] + C$

$\displaystyle = \boxed{\frac{1}{16}\left[\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right] + C}$

2(a) $\displaystyle\int_{-\infty}^{5} \frac{1}{x^2-4x+7}\,dx = \lim_{t\to-\infty}\int_t^5 \frac{1}{(x-2)^2+3}\,dx = \lim_{t\to-\infty}\int_{t-2}^{3} \frac{1}{w^2+3}\,dw$

x^2-4x+4

$\boxed{\begin{array}{l} w = x-2 \\ dw = dx \end{array}}$

$\boxed{\begin{array}{l} x=t \Rightarrow w=t-2 \\ x=5 \Rightarrow w=3 \end{array}}$

$\displaystyle = \lim_{t\to-\infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{w}{\sqrt{3}}\right)\Bigg|_{t-2}^{3}$

$\displaystyle = \lim_{t\to-\infty} \frac{1}{\sqrt{3}}\left[\overset{\pi/3}{\arctan}\left(\overset{\sqrt{3}}{\frac{3}{\sqrt{3}}}\right) - \overset{-\pi/2}{\arctan}\left(\overset{-\infty}{\frac{t-2}{\sqrt{3}}}\right)\right]$

$\displaystyle = \frac{1}{\sqrt{3}}\left[\frac{\pi}{3} + \frac{\pi}{2}\right] = \frac{1}{\sqrt{3}}\left[\frac{5\pi}{6}\right] = \boxed{\frac{5\pi}{6\sqrt{3}}}$ Converges.

$\times(1/x)$

2(b) $\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x \cdot 1\,dx = \lim_{t\to 0^+} x\ln x\Big|_t^1 - \int_t^1 1\,dx$

IBP

$\boxed{\begin{array}{ll} u = \ln x & dv = 1dx \\ du = \frac{1}{x}dx & v = x \end{array}}$

$= \lim_{t\to 0^+} x\ln x\Big|_t^1 - x\Big|_t^1$

don't drop

$= \lim_{t\to 0^+} \underbrace{1\cdot\cancel{\ln 1}}_{0} - \underbrace{\cancel{t\ln t}}_{0\cdot(-\infty)\ (*)\ \to 0} - (1 - \underbrace{\cancel{t}}_{0})$

$= \boxed{-1}$ Converges

(*) $\lim_{t\to 0^+} \overset{0\cdot(-\infty)}{t\ln t} = \lim_{t\to 0^+} \frac{\ln t}{1/t} \overset{-\infty/\infty}{\underset{L'H}{=}} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-1/t^2} \overset{-t}{(-t^2)} = \lim_{t\to 0^+} -t = 0$

t^{-1}

3(a) $1-e+\frac{e^2}{2!}-\frac{e^3}{3!}+\frac{e^4}{4!}-\ldots = \boxed{e^{-e}}$

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

(b) $\sum_{n=0}^{\infty} \frac{8}{5^n} = 8\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n = 8\left[\frac{1}{1-\frac{1}{5}}\right] = 8\left[\frac{1}{\left(\frac{4}{5}\right)}\right] = \cancel{8}^{2}\left(\frac{5}{\cancel{4}}\right) = \boxed{10}$

(c) $-\frac{1}{2e^2}+\frac{1}{3e^3}-\frac{1}{4e^4}+\frac{1}{5e^5}-\ldots = -\frac{\left(\frac{1}{e}\right)^2}{2}+\frac{\left(\frac{1}{e}\right)^3}{3}-\frac{\left(\frac{1}{e}\right)^4}{4}+\ldots = \boxed{\ln\left(1+\frac{1}{e}\right)-\frac{1}{e}}$

$\ln(1+x) = \underbrace{x}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

✓ missing $\frac{1}{e}$

OR $= \boxed{\ln\left(\frac{e+1}{e}\right)-\frac{1}{e}}$

OR $\boxed{\ln(e+1)-\cancel{\ln e}^{\,1}-\frac{1}{e}}$

(d) $\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{9^n(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{3^{2n}(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{3}\right)^{2n}}{(2n+1)!}\frac{\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3}\right)}$

$= \frac{3}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \frac{3}{\pi}\,\cancel{\sin\left(\frac{\pi}{3}\right)}^{\sqrt{3}/2} = \boxed{\frac{3\sqrt{3}}{2\pi}}$

(e) $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n+1}}{9\,(2n)!} = \frac{-\pi}{9}\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!} = -\frac{\pi}{9}\,\cancel{\cos(\pi)}^{-1} = \boxed{\frac{\pi}{9}}$

(f) $4+4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\ldots = \underbrace{4}_{\text{Separate Extra}}+4\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right] = \cancel{4}\,\cancel{\arctan(1)}^{\pi/\cancel{4}}+\underbrace{4}_{\text{extra}} = \boxed{\pi+4}$

Many Options here

4.(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^8+7} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{n^3}{n^8+7} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^8} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ Convergent p-Series $p=5>1$

$$\lim_{n\to\infty} \frac{\frac{n^3}{n^8+7}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{n^8}{n^8+7}\frac{(\frac{1}{n^8})}{(\frac{1}{n^8})} = \lim_{n\to\infty} \frac{1}{1+\frac{7}{n^8}} = 1$$ Finite Non-zero

[OR CT Bound Terms $\frac{n^3}{n^8+7} < \frac{n^3}{n^8} = \frac{1}{n^5}$]

$\Rightarrow$ A.S. Converges by LCT

$\Rightarrow$ O.S. $\boxed{A.C.}$ (by definition)

4(b) $\sum_{n=2}^{\infty} \underbrace{\frac{n^2}{\ln n}}_{a_n}$ Diverges by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{\ln n} = \lim_{x\to\infty} \frac{x^2}{\ln x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{2x}{\frac{1}{x}} = \lim_{x\to\infty} 2x^2 = \infty \neq 0$$

4(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+7} \xrightarrow{A.S.} \sum_{n=0}^{\infty} \frac{1}{n+7} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges p-series (Harmonic) $p=1$

$$\lim_{n\to\infty} \frac{\frac{1}{n+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+7}\frac{(\frac{1}{n})}{(\frac{1}{n})} = \lim_{n\to\infty} \frac{1}{1+\frac{7}{n}} = 1$$ Finite, Non-zero

$\Rightarrow$ A.S. also Diverges by LCT

AST O.S.

① $b_n = \frac{1}{n+7} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+7} = 0$

③ $b_{n+1} = \frac{1}{n+8} < \frac{1}{n+7} = b_n$

Terms Decreasing

O.S. Converges by AST

O.S. $\boxed{C.C}$ (by Definition)

5. $\sum_{n=1}^{\infty} n^n \ln n\,(x-6)^n$

Ratio Test $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}\ln(n+1)(x-6)^{n+1}}{n^n \ln n\,(x-6)^n}\right|$ Must justify L'H Rule.

$$= \lim_{n\to\infty} \underbrace{\frac{(n+1)^n}{n^n}}_{\to e}\underbrace{(n+1)}_{\to\infty}\cdot\underbrace{\frac{\ln(n+1)}{\ln n}}_{\to 1\ \text{See}(*)}|x-6| = \infty > 1$$

Diverges by R.T. unless $x=6$.

$$(*)\ \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} \underset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{x}{x+1} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} 1 = 1$$

6

$$\boxed{\begin{array}{l} I = \{6\} \\ R = 0 \end{array}}$$

6. $\arctan\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \ldots = \frac{1}{2} - \frac{\left(\frac{1}{8}\right)}{3} + \frac{\left(\frac{1}{32}\right)}{5} - \ldots$

$32 \times 5 = 160$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

$$= \frac{1}{2} - \frac{1}{24} + \frac{1}{160} - \ldots$$

$$\approx \frac{1}{2} - \frac{1}{24} = \frac{12}{24} - \frac{1}{24} = \boxed{\frac{11}{24}} \leftarrow \text{Estimate}$$

Using ASET, we can estimate the full Sum using only the first 2 terms with error at Most $\left|1^{st}\text{ Neglected Term}\right| = \frac{1}{160} < \frac{1}{100}$ as desired.

7(a) $\int x^4 e^{-x^3}\,dx = \int x^4 \sum_{n=0}^{\infty}\frac{(-x^3)^n}{n!}\,dx = \int x^4 \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{n!}\,dx$

$$= \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+4}}{n!}\,dx = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+5}}{n!\,(3n+5)} + C}$$

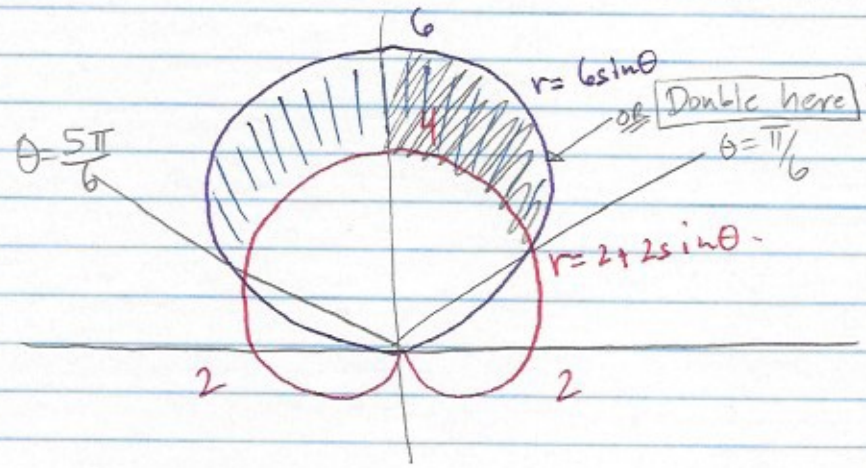
7(b) $$\frac{d}{dx}\left[x^3\sin(6x)\right] = \frac{d}{dx}\left[x^3\sum_{n=0}^{\infty}\frac{(-1)^n(6x)^{2n+1}}{(2n+1)!}\right]$$

$$= \frac{d}{dx}\left[x^3\sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n+1}x^{2n+1}}{(2n+1)!}\right]$$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n+1}x^{2n+4}}{(2n+1)!}\right]$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n+1}(2n+4)x^{2n+3}}{(2n+1)!}}$$

8(a)



Intersect?

$2+2\sin\theta=6\sin\theta$

$4\sin\theta=2$

$\sin\theta=\frac{1}{2}$

$\theta=\pi/6,\ 5\pi/6$

$$A=\frac{1}{2}\int_{\pi/6}^{5\pi/6}(\text{Outer Radius})^2-(\text{Inner Radius})^2\,d\theta$$

$$=\boxed{\frac{1}{2}\int_{\pi/6}^{5\pi/6}(6\sin\theta)^2-(2+2\sin\theta)^2\,d\theta}$$

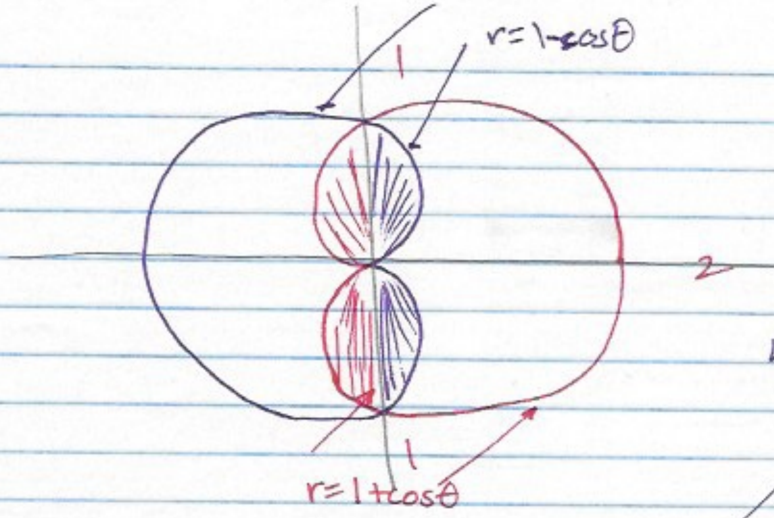
~~OR~~

Double using Symmetry

$$=\boxed{2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2}(6\sin\theta)^2-(2+2\sin\theta)^2\,d\theta\right]}$$

other options as well

8(b)



4 x 

$$\text{Area} = 4\left[\frac{1}{2}\int_0^{\pi/2} (\text{polar radius})^2 \, d\theta\right]$$

$$= \boxed{4\left[\frac{1}{2}\int_0^{\pi/2} (1-\cos\theta)^2 \, d\theta\right]}$$

OR $4\left[\frac{1}{2}\int_{\pi/2}^{\pi} (1+\cos\theta)^2 \, d\theta\right]$ 

OR $2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2} (1-\cos\theta)^2 \, d\theta\right]$ 

OR $2\left[\frac{1}{2}\int_{\pi/2}^{3\pi/2} (1+\cos\theta)^2 \, d\theta\right]$ 

⋮

more?!