Comparison Test

Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with positive terms.

1. If $a_n \leq b_n$ and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $a_n \geq b_n$ and if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

USED: When your given series behaves more like a simpler series, when n is large. Usually the comparison series is taken naturally as a p-series or a geometric series. Also only used for positive termed series.

USED: When you have a direct, quick, obvious and *helpful* bound. Meaning use this test in the two following helpful settings

- **Smaller than a convergent series is convergent.**
- **Larger than a divergent series is divergent.**

WARNING: Be careful making a size argument in the wrong direction. Sometimes you have bounds that are true, but they are not helpful and do not necessarily get you any implication about convergence.

- Smaller than a divergent series **Says Nothing!!!**
- Larger than a convergent series **Says Nothing!!!**

APPROACH:

- Given the original series, start by ignoring non-dominant terms and decide what the comparison series will be. Again, this comparison series is usually a p-series or a geometric series.
- Bound the terms of the two series. Compare the terms of the given series with the terms of the comparision series ("that you like better"). DO NOT PUT sum signs here. Bound only the terms.
- Analyze the comparison series completely.
- Make a conclusion about the original series.

EXAMPLES: Determine and state whether each of the following series **converges** or **diverges**. Name any convergence test(s) that you use, and justify all of your work.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^5+6}$

Note that as n gets large,

$$\sum_{n=1}^{\infty} \frac{n^2}{n^5+6} \approx \sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Bound the terms $\dfrac{n^2}{n^5+6} \le \dfrac{n^2}{n^5} = \dfrac{1}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series with $p = 3 > 1$. Therefore, the orginal series also $\boxed{\text{converges}}$ by CT.

Let's model the next examples a bit more simply. Use as little supporting detail as necessary. Use some shorthand notation as well.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{7+9^n} \approx \sum_{n=1}^{\infty} \frac{1}{9^n}$

Bound the terms $\dfrac{1}{7+9^n} \le \dfrac{1}{9^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{9^n}$ is a Conv. Geom. Series with $|r| = \left|\dfrac{1}{9}\right| = \dfrac{1}{9} < 1$. Then O.S. $\boxed{\text{Converges}}$ by CT.

3. $\displaystyle\sum_{n=4}^{\infty} \frac{n}{n\sqrt{n}-1} \approx \sum_{n=4}^{\infty} \frac{n}{n\sqrt{n}} = \sum_{n=4}^{\infty} \frac{1}{\sqrt{n}}$

Bound the terms $\dfrac{n}{n\sqrt{n}-1} \ge \dfrac{n}{n\sqrt{n}} = \dfrac{1}{\sqrt{n}}$ and $\displaystyle\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a Divergent p-Series with $p = \dfrac{1}{2} < 1$. Then O.S. $\boxed{\text{Div.}}$ by CT.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^5+3} \approx \sum_{n=1}^{\infty} \frac{1}{n^5}$

Bound the terms $\dfrac{\sin^2 n}{n^5+3} \le \dfrac{1}{n^5+3} \le \dfrac{1}{n^5}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ is a Conv. p-Series with $p = 5 > 1$. Then O.S. $\boxed{\text{Conv.}}$ by CT.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^8+1} \approx \sum_{n=1}^{\infty} \frac{1}{n^8}$

Bound the terms $\dfrac{\arctan n}{n^8+1} \le \dfrac{\frac{\pi}{2}}{n^8+1} \le \left(\dfrac{\pi}{2}\right)\dfrac{1}{n^8}$ and $\left(\dfrac{\pi}{2}\right)\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^8}$ is a Constant Multiple of a Conv. p-Series with $p = 8 > 1$ which is therefore Conv. Then O.S. $\boxed{\text{Conv.}}$ by CT.

Note here: If you want to ignore the $\dfrac{\pi}{2}$, the the Direct Comparison Test (CT) will not work easily here, because $\dfrac{\left(\frac{\pi}{2}\right)}{n^8}$ is NOT less than or equal to $\dfrac{1}{n^8}$ because $\dfrac{\pi}{2} > 1$.