

Absolute Convergence Test

Given a series $\sum_{n=1}^{\infty} a_n$, if the Absolute Series $\sum_{n=1}^{\infty} |a_n|$ converges, then the Original Series $\sum_{n=1}^{\infty} a_n$ converges.

USED: When the Absolute Series is easier to analyze.

USED: To avoid analyzing negative signs, or maybe the Alternating Series Test.

BENEFITS: Handling series with positive terms is usually easier. Plus, you have more convergence tests available for positive termed series; think IT, CT, LCT.

WARNING: If you analyze the Absolute Series, and it is NOT convergent, then this convergence test is inconclusive. It does not immediately follow up with a conclusion about Conditional Convergence without more work. All you know if the the A.S. diverges is that your original series is NOT Absolutely Convergent (A.C.)

NOTE: If the Original Series is already equal to the Absolute series, then Absolute Convergence is the same as Convergence.

NOTE: If the Original Series is not the same as the Absolute Series, then this ACT says that **Absolute Convergence implies Convergence**.

APPROACH:

- Given the original series, find the Absolute Series by ignoring any negative signs.
- Analyze the Absolute Series completely as before. Run all the details of p -Series Test, IT, CT, or LCT. Justify everything. Make a clear conclusion about the convergence of the Absolute Series.
- If the A.S. converges, then you are done, and the O.S. converges by ACT.
- If the A.S. diverges, you have no conclusion about the O.S. You are stuck running a convergence test (likely AST) on the O.S.

EXAMPLES: Determine and state whether each of the following series **converges** or **diverges**. Name any convergence test(s) that you use, and justify all of your work.

1. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^5} \rightarrow$ Check the Absolute Series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ which is a convergent p -Series with $p = 5 > 1$.

Then the Original Series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^5}$ is convergent by ACT.

NOTE: this technique helps you avoid AST here on the O.S.

2. $\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2 n}{n^3 + 5} \rightarrow$ Consider the A.S. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3 + 5}$.

Bound the terms: $\frac{\sin^2 n}{n^3 + 5} \leq \frac{\sin^2 n}{n^3} \leq \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p -Series with $p = 3 > 1$.

Therefore, the A.S. is convergent by CT. Finally, the O.S. is convergent by ACT.

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3 + 5n}{n^8 + 7} \rightarrow$ A.S. $\sum_{n=1}^{\infty} \frac{n^3 + 5n}{n^8 + 7} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^8} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ Conv. p -series $p = 5 > 1$.

Check: $\lim_{n \rightarrow \infty} \frac{\frac{n^3 + 5n}{n^8 + 7}}{\frac{1}{n^5}} = \lim_{n \rightarrow \infty} \frac{n^8 + 5n^6}{n^8 + 7} = \lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n^2}}{1 + \frac{7}{n^8}} = 1$ Finite, Non-zero.

Therefore the A.S. is also convergent by LCT (which by definition means that the A.S. is A.C.). Finally, the O.S. is convergent by ACT.

4. $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan n}{n^2 + 9} \rightarrow$ A.S. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2 + 9} \approx \sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ Constant Multiple of a Convergent p -series $p = 2 > 1$ is Convergent.

Bound the terms: $\frac{\arctan n}{n^2 + 9} \leq \frac{\frac{\pi}{2}}{n^2 + 9} \leq \frac{\frac{\pi}{2}}{n^2}$. We analyzed the comparison series above.

Finally, the A.S. is convergent by CT which implies that the O.S. is convergent by ACT.

LAST NOTE: It is very important to finish the full and complete analysis of the Absolute Series. Make a conclusion about the convergence of the Absolute Series. THEN you can use that to imply convergence of the O.S. using the ACT.