

***u*-substitution technique for Integration**

Before this Math 121 course, we have three main techniques of Integration.

1. *We know it* base Snap facts (with single variables)
2. Algebra, FOIL or *split-split* algebra
3. *u*-substitution

The technique of *u*-substitution is a temporary convenience that essentially **reverses the Chain Rule**.

Example: The Chain Rule yields

$$\frac{d}{dx} \sin(x^3) = 3x^2 \cos(x^3) \quad \text{which gives } \int 3x^2 \cos(x^3) dx = \sin(x^3) + C$$

Q: How can we compute these complicated integrals with *nested* pieces?

- The substitution method *hides a nested* part of your integrand and aims to match the derivative piece at about the same time.
- We need to choose *u* to be a nested chunk of your integrand, pretty much a grab-of-sorts of the inside portion of a composed function.
- Once you choose *u* as some hidden chunk of your integrand, that will yield a certain derivative *du*. In the end, we want to choose a substitution *u* that simplifies the Integral **and** also matches a part as the derivative.

$$\int \underbrace{f'(g(x))}_u \cdot \underbrace{g'(x) dx}_{du} = \int f'(u) du = f(u) + C = f(g(x)) + C$$

where

$$\begin{aligned} u &= g(x) \\ du &= g'(x) dx \end{aligned}$$

INDEFINITE Integrals: Always remember to add +C right away , as soon as you compute the Most General Antiderivative. The original variable always reappears when we re-substitute back for *u*.

$$\text{Ex: } \int \underbrace{x^6 \left(\underbrace{x^7 - 9}_u \right)^8}_{\frac{1}{7} du} dx = \frac{1}{7} \int u^8 du = \frac{1}{7} \left(\frac{u^9}{9} \right) + C = \frac{(x^7 - 9)^9}{63} + C$$

$$\begin{aligned} u &= x^7 - 9 \\ du &= 7x^6 dx \\ \frac{1}{7} du &= x^6 dx \end{aligned}$$

$$\text{Ex: } \int \sin(\underbrace{6x}_u) \underbrace{dx}_{\frac{1}{6}du} = \frac{1}{6} \int \sin u \, du = \frac{1}{6} (-\cos u) + C = \boxed{-\frac{1}{6} \cos(6x) + C}$$

$$\begin{aligned} u &= 6x \\ du &= 6 \, dx \\ \frac{1}{6}du &= dx \end{aligned}$$

$$\text{Ex: } \int \frac{\overbrace{\sec^2 x}^{du}}{\underbrace{\sqrt{5 + \tan x}}_u} \underbrace{dx}_{\frac{1}{6}du} = \int \frac{1}{\sqrt{u}} \, du = \int u^{-\frac{1}{2}} \, du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{u} + C = \boxed{2\sqrt{5 + \tan x} + C}$$

$$\begin{aligned} u &= 5 + \tan x \\ du &= \sec^2 x \, dx \end{aligned}$$

$$\begin{aligned} \text{Ex: } \int \frac{5}{x^2 \left(8 + \frac{2}{x}\right)^3} dx &= -\frac{5}{2} \int \frac{1}{u^3} \, du = -\frac{5}{2} \int u^{-3} \, du = -\frac{5}{2} \left(\frac{u^{-2}}{-2} \right) + C \\ &= \frac{5}{4u^2} + C = \boxed{\frac{5}{4 \left(8 + \frac{2}{x}\right)^2} + C} \end{aligned}$$

$$\begin{aligned} u &= 8 + \frac{2}{x} \\ du &= -\frac{2}{x^2} \, dx \\ -\frac{1}{2}du &= \frac{1}{x^2} \, dx \end{aligned}$$

$$\begin{aligned} \text{Ex: } \int \frac{7}{\sqrt{x} \left(3 + \sqrt{x}\right)^2} dx &= 7 \int \frac{1}{\sqrt{x} (3 + \sqrt{x})^2} dx = 14 \int \frac{1}{u^2} \, du = 14 \int u^{-2} \, du \\ &= 14 \left(\frac{u^{-1}}{-1} \right) + C = -\frac{14}{u} + C = \boxed{-\frac{14}{3 + \sqrt{x}} + C} \end{aligned}$$

$$\begin{aligned} u &= 3 + \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}} \, dx \\ 2du &= \frac{1}{\sqrt{x}} \, dx \end{aligned}$$

DEFINITE Integrals: Recall, you must change (or temporarily mark) your Limits of integration. The variables and Limits of Integration change *simultaneously*. Once you *switch* your Limits of Integration to u -values, then the original variable never reappears.

Ex:

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^3 x} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{(\cos x)^3} dx = - \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{u^3} du = - \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} u^{-3} du = - \left(\frac{u^{-2}}{-2} \right) \bigg|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} = \frac{1}{2u^2} \bigg|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \\ &= \frac{1}{2 \left(\frac{1}{2}\right)^2} - \frac{1}{2 \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2 \left(\frac{1}{4}\right)} - \frac{1}{2 \left(\frac{3}{4}\right)} = \frac{1}{\frac{1}{2}} - \frac{1}{\frac{3}{2}} = 2 - \frac{2}{3} = \frac{6}{3} - \frac{2}{3} = \boxed{\frac{4}{3}}\end{aligned}$$

$\begin{aligned}u &= \cos x \\ du &= -\sin x \, dx \\ -du &= \sin x \, dx\end{aligned}$	and	$\begin{aligned}x = \frac{\pi}{6} &\Rightarrow u = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\ x = \frac{\pi}{3} &\Rightarrow u = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}\end{aligned}$
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OR here is an **ALTERNATE** option if you do not want to *Change* your Limits of Integration to u -limits. If you opt to *Mark* your Limits of Integration instead of *Changing* them to u Limits, then the original variable does reappear. Be careful not to mix and match x and u pieces.

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^3 x} dx &= - \int_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} \frac{1}{u^3} du = - \int_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} u^{-3} du = - \left(\frac{u^{-2}}{-2} \right) \bigg|_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} = \frac{1}{2u^2} \bigg|_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} \\ &= \frac{1}{2 \cos^2 x} \bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{2 \left(\cos\left(\frac{\pi}{3}\right)\right)^2} - \frac{1}{2 \left(\cos\left(\frac{\pi}{6}\right)\right)^2} = \dots = \boxed{\frac{4}{3}}\end{aligned}$$

Note: Same u -sub as above, and same final values ...

Ex:

$$\begin{aligned}\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(3x) dx &= \frac{1}{3} \int_{-\pi}^{\frac{3\pi}{2}} \sin u \, du = -\frac{1}{3} \cos u \bigg|_{-\pi}^{\frac{3\pi}{2}} \\ &= -\frac{1}{3} \cos\left(\frac{3\pi}{2}\right) - \left(-\frac{1}{3} \cos(-\pi)\right) = -0 + \frac{1}{3}(-1) = \boxed{-\frac{1}{3}}\end{aligned}$$

$\begin{aligned}u &= 3x \\ du &= 3 \, dx \\ \frac{1}{3}du &= dx\end{aligned}$	and	$\begin{aligned}x = -\frac{\pi}{3} &\Rightarrow u = 3\left(-\frac{\pi}{3}\right) = -\pi \\ x = \frac{\pi}{2} &\Rightarrow u = 3\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}\end{aligned}$
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Here are some examples of *inverted* or *reverse* substitutions. When using a u -substitution, we are fixing a (temporary) relationship between x and u for the entire problem. So, if there are any extra x variable leftover after the standard u -substitution, then you can solve the original choice of u in terms of x instead for x in terms of u . Then substitute that in for any leftover x 's and then continue on with the antiderivative, etc.

Ex:

$$\int x \sqrt{x+1} \, dx = \int (u-1) \sqrt{u} \, du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}} \, du = \frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \boxed{\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C}$$

$$\boxed{\begin{array}{l} u = x+1 \Rightarrow x = u-1 \\ du = 1 \, dx \end{array}}$$

Ex:

$$\int \frac{x}{\sqrt{3-x}} \, dx = - \int \frac{3-u}{\sqrt{u}} \, du = - \int \frac{3}{\sqrt{u}} - \frac{u}{\sqrt{u}} \, du = - \int 3u^{-\frac{1}{2}} - u^{\frac{1}{2}} \, du$$

$$= - \left(3 \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right) + C = -6\sqrt{u} + \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \boxed{-6\sqrt{3-x} + \frac{2}{3}(3-x)^{\frac{3}{2}} + C}$$

$$\boxed{\begin{array}{l} u = 3-x \Rightarrow x = 3-u \\ du = -1 \, dx \\ -du = dx \end{array}}$$