

## ***u*-substitution technique for Integration**

Before this Math 121 course, we have three main techniques of Integration.

1. *We know it* base Snap facts (with single variables)
2. Algebra, FOIL or *split-split* algebra
3. *u*-substitution

The technique of *u*-substitution is a temporary convenience that essentially **reverses the Chain Rule**.

Example: The Chain Rule yields

$$\frac{d}{dx} \sin(x^3) = 3x^2 \cos(x^3) \quad \text{which gives } \int 3x^2 \cos(x^3) \, dx = \sin(x^3) + C$$

Q: How can we compute these complicated integrals with *nested* pieces?

- The substitution method *hides a nested* part of your integrand and aims to match the derivative piece at about the same time.
- We need to choose  $u$  to be a nested chunk of your integrand, pretty much a grab-of-sorts of the inside portion of a composed function.
- Once you choose  $u$  as some hidden chunk of your integrand, that will yield a certain derivative  $du$ . In the end, we want to choose a substitution  $u$  that simplifies the Integral **and** also matches a part as the derivative.

$$\int \underbrace{f'(g(x))}_{u} \cdot \underbrace{g'(x)}_{du} \, dx = \int f'(u) \, du = f(u) + C = f(g(x)) + C$$

where

$u$	$= g(x)$
$du$	$= g'(x) \, dx$

**INDEFINITE Integrals:** Always remember to add  $+C$  right away, as soon as you compute the Most General Antiderivative. The original variable always reappears when we re-substitute back for  $u$ .

Ex: 
$$\int \underbrace{x^6}_{\frac{1}{7}du} \left( \underbrace{x^7 - 9}_{u} \right)^8 \, dx = \frac{1}{7} \int u^8 \, du = \frac{1}{7} \left( \frac{u^9}{9} \right) + C = \boxed{\frac{(x^7 - 9)^9}{63} + C}$$

$u$	$= x^7 - 9$
$du$	$= 7x^6 \, dx$
$\frac{1}{7}du$	$= x^6 \, dx$

Ex:  $\int \sin(\underbrace{6x}_u) \underbrace{dx}_{\frac{1}{6}du} = \frac{1}{6} \int \sin u \, du = \frac{1}{6} (-\cos u) + C = \boxed{-\frac{1}{6} \cos(6x) + C}$

$u = 6x$
$du = 6 \, dx$
$\frac{1}{6}du = dx$

Ex:  $\int \frac{\overbrace{\sec^2 x}^{du}}{\sqrt{\underbrace{5 + \tan x}_u}} \, dx = \int \frac{1}{\sqrt{u}} \, du = \int u^{-\frac{1}{2}} \, du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{u} + C = \boxed{2\sqrt{5 + \tan x} + C}$

$u = 5 + \tan x$
$du = \sec^2 x \, dx$

Ex:  $\int \frac{5}{x^2 \left(8 + \frac{2}{x}\right)^3} \, dx = -\frac{5}{2} \int \frac{1}{u^3} \, du = -\frac{5}{2} \int u^{-3} \, du = -\frac{5}{2} \left(\frac{u^{-2}}{-2}\right) + C$

$$= \frac{5}{4u^2} + C = \boxed{\frac{5}{4\left(8 + \frac{2}{x}\right)^2} + C}$$

$u = 8 + \frac{2}{x}$
$du = -\frac{2}{x^2} \, dx$
$-\frac{1}{2}du = \frac{1}{x^2} \, dx$

Ex:  $\int \frac{7}{\sqrt{x} \left(3 + \sqrt{x}\right)^2} \, dx = 7 \int \frac{1}{\sqrt{x} (3 + \sqrt{x})^2} \, dx = 14 \int \frac{1}{u^2} \, du = 14 \int u^{-2} \, du$

$$= 14 \left(\frac{u^{-1}}{-1}\right) + C = -\frac{14}{u} + C = \boxed{-\frac{14}{3 + \sqrt{x}} + C}$$

$u = 3 + \sqrt{x}$
$du = \frac{1}{2\sqrt{x}} \, dx$
$2du = \frac{1}{\sqrt{x}} \, dx$

**DEFINITE Integrals:** Recall, you must change (or temporarily mark) your Limits of integration. The variables and Limits of Integration change *simultaneously*. Once you *switch* your Limits of Integration to  $u$ -values, then the original variable never reappears.

Ex:

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^3 x} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{(\cos x)^3} dx = - \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{u^3} du = - \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} u^{-3} du = - \left( \frac{u^{-2}}{-2} \right) \Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} = \frac{1}{2u^2} \Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \\ &= \frac{1}{2 \left( \frac{1}{2} \right)^2} - \frac{1}{2 \left( \frac{\sqrt{3}}{2} \right)^2} = \frac{1}{2 \left( \frac{1}{4} \right)} - \frac{1}{2 \left( \frac{3}{4} \right)} = \frac{1}{\frac{1}{2}} - \frac{1}{\frac{3}{2}} = 2 - \frac{2}{3} = \frac{6}{3} - \frac{2}{3} = \boxed{\frac{4}{3}} \end{aligned}$$

$u = \cos x$ $du = -\sin x dx$ $-du = \sin x dx$	and <table border="1" style="display: inline-table; vertical-align: middle;"> <tr> <td style="padding: 5px;"> <math>x = \frac{\pi}{6} \Rightarrow u = \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}</math>  <math>x = \frac{\pi}{3} \Rightarrow u = \cos \left( \frac{\pi}{3} \right) = \frac{1}{2}</math> </td> </tr> </table>	$x = \frac{\pi}{6} \Rightarrow u = \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}$ $x = \frac{\pi}{3} \Rightarrow u = \cos \left( \frac{\pi}{3} \right) = \frac{1}{2}$
$x = \frac{\pi}{6} \Rightarrow u = \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}$ $x = \frac{\pi}{3} \Rightarrow u = \cos \left( \frac{\pi}{3} \right) = \frac{1}{2}$		

OR here is an **ALTERNATE** option if you do not want to *Change* your Limits of Integration to  $u$ -limits. If you opt to *Mark* your Limits of Integration instead of *Changing* them to  $u$  Limits, then the original variable does reappear. Be careful not to mix and match  $x$  and  $u$  pieces.

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^3 x} dx &= - \int_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} \frac{1}{u^3} du = - \int_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} u^{-3} du = - \left( \frac{u^{-2}}{-2} \right) \Big|_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} = \frac{1}{2u^2} \Big|_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} \\ &= \frac{1}{2 \cos^2 x} \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{2 \left( \cos \left( \frac{\pi}{3} \right) \right)^2} - \frac{1}{2 \left( \cos \left( \frac{\pi}{6} \right) \right)^2} = \dots = \boxed{\frac{4}{3}} \end{aligned}$$

Note: Same  $u$ -sub as above, and same final values . . .

Ex:

$$\begin{aligned} \int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(3x) dx &= \frac{1}{3} \int_{-\pi}^{\frac{3\pi}{2}} \sin u du = -\frac{1}{3} \cos u \Big|_{-\pi}^{\frac{3\pi}{2}} \\ &= -\frac{1}{3} \cos \left( \frac{3\pi}{2} \right) - \left( -\frac{1}{3} \cos(-\pi) \right) = -0 + \frac{1}{3}(-1) = \boxed{-\frac{1}{3}} \end{aligned}$$

$u = 3x$ $du = 3 dx$ $\frac{1}{3}du = dx$	and <table border="1" style="display: inline-table; vertical-align: middle;"> <tr> <td style="padding: 5px;"> <math>x = -\frac{\pi}{3} \Rightarrow u = 3 \left( -\frac{\pi}{3} \right) = -\pi</math>  <math>x = \frac{\pi}{2} \Rightarrow u = 3 \left( \frac{\pi}{2} \right) = \frac{3\pi}{2}</math> </td> </tr> </table>	$x = -\frac{\pi}{3} \Rightarrow u = 3 \left( -\frac{\pi}{3} \right) = -\pi$ $x = \frac{\pi}{2} \Rightarrow u = 3 \left( \frac{\pi}{2} \right) = \frac{3\pi}{2}$
$x = -\frac{\pi}{3} \Rightarrow u = 3 \left( -\frac{\pi}{3} \right) = -\pi$ $x = \frac{\pi}{2} \Rightarrow u = 3 \left( \frac{\pi}{2} \right) = \frac{3\pi}{2}$		

Here are some examples of *inverted* or *reverse* substitutions. When using a  $u$ -substitution, we are fixing a (temporary) relationship between  $x$  and  $u$  for the entire problem. So, if there are any extra  $x$  variable leftover after the standard  $u$ -substitution, then you can solve the original choice of  $u$  in terms of  $x$  instead for  $x$  in terms of  $u$ . Then substitute that in for any leftover  $x$ 's and then continue on with the antiderivative, etc.

Ex:

$$\int x\sqrt{x+1} dx = \int (u-1)\sqrt{u} du = \int u^{\frac{3}{2}} - u^{\frac{1}{2}} du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \boxed{\frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C}$$

$$\boxed{\begin{aligned} u &= x+1 \Rightarrow x = u-1 \\ du &= 1 dx \end{aligned}}$$

Ex:

$$\int \frac{x}{\sqrt{3-x}} dx = - \int \frac{3-u}{\sqrt{u}} du = - \int \frac{3}{\sqrt{u}} - \frac{u}{\sqrt{u}} du = - \int 3u^{-\frac{1}{2}} - u^{\frac{1}{2}} du$$

$$= - \left( 3 \left( \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right) + C = -6\sqrt{u} + \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \boxed{-6\sqrt{3-x} + \frac{2}{3}(3-x)^{\frac{3}{2}} + C}$$

$$\boxed{\begin{aligned} u &= 3-x \Rightarrow x = 3-u \\ du &= -1 dx \\ -du &= dx \end{aligned}}$$