

Extra Examples of Trigonometric Substitutions

Math 121 D. Benedetto

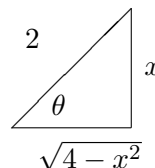
1.

$\int \frac{x^2}{\sqrt{4-x^2}} dx$	Recognize difference of squares under the square root
$= \int \frac{(2 \sin \theta)^2}{\sqrt{4-4 \sin^2 \theta}} 2 \cos \theta d\theta$	Use a sine sub for $a^2 - x^2$. Remember, sub for dx
$= \int \frac{4 \sin^2 \theta}{\sqrt{4(1-\sin^2 \theta)}} 2 \cos \theta d\theta$	Work the algebra to create the identity $1 - \sin^2 \theta = \cos^2 \theta$
$= 4 \int \frac{\sin^2 \theta}{\sqrt{4} \sqrt{\cos^2 \theta}} 2 \cos \theta d\theta$	The identity creates the perfect square under the root
$= 4 \int \frac{\sin^2 \theta}{2 \cos \theta} 2 \cos \theta d\theta$	Simplify the root of the perfect square, see what cancels
$= 4 \int \sin^2 \theta d\theta$	Prepare for the EVEN power technique
$= 4 \int \frac{1 - \cos(2\theta)}{2} d\theta$	Using the half angle identity
$= 2 \int 1 - \cos(2\theta) d\theta$	Using a u-sub if needed
$= 2 \left(\theta - \frac{\sin(2\theta)}{2} \right) + C$	
$= 2 \left(\theta - \frac{2 \sin \theta \cos \theta}{2} \right) + C$	Using the Double Angle Identity
$= 2(\theta - \sin \theta \cos \theta) + C$	Using the Trig Triangle to <i>unwind</i> the Trig Sub
$= 2 \left(\arcsin \left(\frac{x}{2} \right) - \left(\frac{x}{2} \right) \left(\frac{\sqrt{4-x^2}}{2} \right) \right) + C = \boxed{2 \arcsin \left(\frac{x}{2} \right) - \left(\frac{x \sqrt{4-x^2}}{2} \right) + C}$	

Trig. Substitute

$$x = 2 \sin \theta \Rightarrow \sin \theta = \frac{x}{2}$$

$$dx = 2 \cos \theta d\theta$$



2. Try the same problem in a definite integral. You can either change your limits with the trig sub **or** mark them as x limits. You should choose only one method below.

Option 1: Changing Limits

$$\begin{aligned}
 \int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(2\sin\theta)^2}{\sqrt{4-4\sin^2\theta}} 2\cos\theta d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}} 2\cos\theta d\theta \\
 &= 4 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2\theta}{\sqrt{4}\sqrt{\cos^2\theta}} 2\cos\theta d\theta = 4 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2\theta}{2\cos\theta} 2\cos\theta d\theta = 4 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta d\theta \\
 &= 4 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos(2\theta)}{2} d\theta = 2 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1-\cos(2\theta) d\theta = 2 \left(\theta - \frac{\sin(2\theta)}{2} \right) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
 &= 2 \left(\left(\frac{\pi}{3} - \frac{\sin\left(\frac{2\pi}{3}\right)}{2} \right) - \left(\frac{\pi}{6} - \frac{\sin\left(\frac{\pi}{3}\right)}{2} \right) \right) = 2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\sqrt{3}}{4} \right) = \boxed{\frac{\pi}{3}}
 \end{aligned}$$

Recall: $x = 2\sin\theta \Rightarrow \theta = \arcsin\left(\frac{x}{2}\right)$.

Change Limits:

$$\begin{aligned}
 x = 1 &\Rightarrow \theta = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \\
 x = \sqrt{3} &\Rightarrow \theta = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}
 \end{aligned}$$

Note: If you change your limits, you do not need the Trig Triangle to return to x -variable.

Option 2: Mark Limits

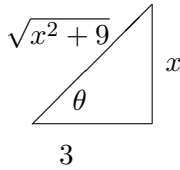
$$\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}} dx = \int_{x=1}^{x=\sqrt{3}} \frac{(2\sin\theta)^2}{\sqrt{4-4\sin^2\theta}} 2\cos\theta d\theta = \dots = 2(\theta - \sin\theta\cos\theta) \Big|_{x=1}^{x=\sqrt{3}}$$

using antiderivative from above in 1.

$$\begin{aligned}
 &= 2\arcsin\left(\frac{x}{2}\right) - \left(\frac{x\sqrt{4-x^2}}{2}\right) \Big|_1^{\sqrt{3}} = 2\arcsin\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}\sqrt{4-3}}{2}\right) - \left[2\arcsin\left(\frac{1}{2}\right) - \left(\frac{1\sqrt{4-1}}{2}\right)\right] \\
 &= 2\left(\frac{\pi}{3}\right) - \frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} = 2\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 2\left(\frac{\pi}{6}\right) = \boxed{\frac{\pi}{3}}
 \end{aligned}$$

3.

$$\begin{aligned}
 & \int \frac{1}{(9+x^2)^{\frac{5}{2}}} dx && \text{Recognize **sum** of squares under the square root} \\
 &= \int \frac{1}{\left(\sqrt{9+9\tan^2\theta}\right)^5} 3\sec^2\theta d\theta && \text{Use a tangent sub for } a^2+x^2. \text{ **Remember**, sub for } dx \\
 &= \int \frac{1}{\left(\sqrt{9(1+\tan^2\theta)}\right)^5} 3\sec^2\theta d\theta && \text{Work the algebra to create the identity } 1+\tan^2\theta=\sec^2\theta \\
 &= \int \frac{1}{\left(\sqrt{9}\sqrt{\sec^2\theta}\right)^5} 3\sec^2\theta d\theta && \text{The identity creates the perfect square under the root} \\
 &= \frac{3}{3^5} \int \frac{1}{\sec^5\theta} \sec^2\theta d\theta && \text{Simplify the root of the perfect square, see what cancels} \\
 &= \frac{1}{3^4} \int \frac{1}{\sec^3\theta} d\theta && \text{Flip the secant to cosine} \\
 &= \frac{1}{3^4} \int \cos^3\theta d\theta && \textbf{Prepare for the ODD power technique} \\
 &= \frac{1}{81} \int \cos^2\theta \cos\theta d\theta && \textit{Isolate one copy of cosine} \\
 &= \frac{1}{81} \int (1-\sin^2\theta) \cos\theta d\theta && \textit{Convert using trig identity} \\
 &= \frac{1}{81} \int 1-w^2 dw && \textit{Finish with a substitution} \\
 &= \frac{1}{81} \left(w - \frac{w^3}{3} \right) + C \\
 &= \frac{1}{81} \left(\sin\theta - \frac{\sin^3\theta}{3} \right) + C && = \boxed{\frac{1}{81} \left(\frac{x}{\sqrt{9+x^2}} - \frac{1}{3} \left(\frac{x}{\sqrt{9+x^2}} \right)^3 \right) + C}
 \end{aligned}$$

Trig. Substitute	$ \begin{aligned} x &= 3\tan\theta \Rightarrow \tan\theta = \frac{x}{3} \\ dx &= 3\sec^2\theta d\theta \end{aligned} $		Standard substitution	$ \begin{aligned} w &= \sin\theta \\ dw &= \cos\theta d\theta \end{aligned} $
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$$\begin{aligned}
4. \int x \arcsin x \, dx &= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \, dx \\
&= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cdot \cos \theta \, d\theta = \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{\sin^2 \theta}{\sqrt{\cos^2 \theta}} \cdot \cos \theta \, d\theta \\
&= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{\sin^2 \theta}{\cos \theta} \cdot \cos \theta \, d\theta = \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \sin^2 \theta \, d\theta \\
&= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int \frac{1 - \cos(2\theta)}{2} \, d\theta = \frac{x^2}{2} \arcsin x - \frac{1}{4} \int 1 - \cos(2\theta) \, d\theta \\
&= \frac{x^2}{2} \arcsin x - \frac{1}{4} \left[\theta - \frac{1}{2} \sin(2\theta) \right] + C = \frac{x^2}{2} \arcsin x - \frac{1}{4} \theta + \frac{1}{8} \sin(2\theta) + C \\
&= \frac{x^2}{2} \arcsin x - \frac{1}{4} \theta + \frac{1}{8} (2 \sin \theta \cos \theta) + C = \boxed{\frac{x^2}{2} \arcsin x - \frac{1}{4} \arcsin x + \frac{1}{4} x \sqrt{1-x^2} + C}
\end{aligned}$$

IBP to start:

$ \begin{aligned} u &= \arcsin x & dv &= x \, dx \\ du &= \frac{1}{\sqrt{1-x^2}} \, dx & v &= \frac{x^2}{2} \end{aligned} $

Trig. Substitute

$ \begin{aligned} x &= \sin \theta \\ dx &= \cos \theta \, d\theta \end{aligned} $
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