

Extra Examples of Improper Integrals

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For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

$$1. \int_8^{\infty} \frac{1}{x^2 - 10x + 28} dx = \lim_{t \rightarrow \infty} \int_8^t \frac{1}{x^2 - 10x + 28} dx = \lim_{t \rightarrow \infty} \int_8^t \frac{1}{(x-5)^2 + 3} dx \quad \text{complete the square}$$

Substitute

$u = x - 5$	$x = 8 \Rightarrow u = 3$
$du = dx$	$x = t \Rightarrow u = t - 5$

$$= \lim_{t \rightarrow \infty} \int_3^{t-5} \frac{1}{u^2 + 3} du = \lim_{t \rightarrow \infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right) \Big|_3^{t-5} = \lim_{t \rightarrow \infty} \frac{1}{\sqrt{3}} \left(\arctan\left(\frac{t-5}{\sqrt{3}}\right) - \arctan\left(\frac{3}{\sqrt{3}}\right) \right)$$

$$= \lim_{t \rightarrow \infty} \frac{1}{\sqrt{3}} \left(\arctan\left(\frac{t-5}{\sqrt{3}}\right) - \arctan(\sqrt{3}) \right)$$

$$= \frac{1}{\sqrt{3}} \left(\frac{\pi}{2} - \frac{\pi}{3} \right) = \frac{1}{\sqrt{3}} \left(\frac{3\pi}{6} - \frac{2\pi}{6} \right) = \frac{1}{\sqrt{3}} \left(\frac{\pi}{6} \right) = \boxed{\frac{\pi}{6\sqrt{3}}} \quad \text{Converges}$$

using the formula $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$ and $\lim_{p \rightarrow \infty} \arctan p = \frac{\pi}{2}$

$$2. \int_0^{\frac{1}{2}} \frac{1}{x \ln x} dx = \lim_{t \rightarrow 0^+} \int_t^{\frac{1}{2}} \frac{1}{x \ln x} dx$$

$$= \lim_{t \rightarrow 0^+} \int_{\ln t}^{\ln(\frac{1}{2})} \frac{1}{u} du = \lim_{t \rightarrow 0^+} \ln |\ln x| \Big|_{\ln t}^{\ln(\frac{1}{2})}$$

$$= \lim_{t \rightarrow 0^+} \ln \left| \ln \left(\frac{1}{2} \right) \right| - \lim_{t \rightarrow 0^+} \ln |\ln t| = \ln \left| \ln \left(\frac{1}{2} \right) \right| - \infty = \boxed{-\infty} \quad \text{Diverges}$$

Substitute

$u = \ln x$	$x = t \Rightarrow u = \ln t$
$du = \frac{1}{x} dx$	$x = \frac{1}{2} \Rightarrow u = \ln \left(\frac{1}{2} \right)$

NOTE: $\lim_{t \rightarrow 0^+} \ln t = -\infty$ and then $\lim_{p \rightarrow \infty} \ln p = \infty$

$$\begin{aligned}
3. \int_7^{\infty} \frac{4}{x^2 - 8x + 12} dx &= \lim_{t \rightarrow \infty} \int_7^t \frac{4}{x^2 - 8x + 12} dx \\
&= \lim_{t \rightarrow \infty} \int_7^t \frac{4}{(x-6)(x-2)} dx = \lim_{t \rightarrow \infty} \int_7^t \frac{1}{x-6} - \frac{1}{x-2} dx \quad \text{See PFD below} \\
&= \lim_{t \rightarrow \infty} \ln|x-6| - \ln|x-2| \Big|_7^t = \lim_{t \rightarrow \infty} \ln|t-6| - \ln|t-2| - (\ln|1| - \ln|5|) \\
&\text{(Indeterminate Difference)} \\
&= \lim_{t \rightarrow \infty} \ln \left| \frac{t-6}{t-2} \right| - 0 + \ln 5 = \ln \left| \lim_{t \rightarrow \infty} \frac{t-6}{t-2} \right| + \ln 5 \stackrel{\text{L'H}}{=} \ln \left| \lim_{t \rightarrow \infty} \frac{1}{1} \right| + \ln 5 = 0 + \ln 5 \\
&= \boxed{\ln 5} \text{ Converges}
\end{aligned}$$

Partial Fractions Decomposition:

$$\frac{4}{(x-6)(x-2)} = \frac{A}{x-6} + \frac{B}{x-2}$$

Clearing the denominator yields:

$$\begin{aligned}
4 &= A(x-2) + B(x-6) \\
4 &= (A+B)x - 2A - 6B \\
\text{so that } A+B &= 0, \text{ and } -2A - 6B = 4 \\
\text{Solve for } A &= 1, \text{ and } B = -1
\end{aligned}$$

Note: To avoid the L'H Rule finish, you can also use divides-by-algebra.

$$\lim_{t \rightarrow \infty} \frac{t-6}{t-2} \cdot \frac{\frac{1}{t}}{\frac{1}{t}} = \lim_{t \rightarrow \infty} \frac{1 - \frac{6}{t}}{1 - \frac{2}{t}} = 1$$

$$4. \int_1^2 \frac{4}{x^2 - 8x + 12} dx = \int_1^2 \frac{4}{(x-6)(x-2)} dx = \lim_{t \rightarrow 2^-} \int_1^t \frac{4}{(x-6)(x-2)} dx$$

(See PFD above)

$$\begin{aligned}
&= \lim_{t \rightarrow 2^-} \int_1^t \frac{1}{x-6} - \frac{1}{x-2} dx = \lim_{t \rightarrow 2^-} \ln|x-6| - \ln|x-2| \Big|_1^t \\
&= \lim_{t \rightarrow 2^-} \ln|t-6| - \ln|t-2| - (\ln|-5| - \ln|-1|)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{t \rightarrow 2^-} \ln 4 - \ln|t-2| - \ln 5 + 0 = \ln 4 - (-\infty) - \ln 5 = \boxed{+\infty} \text{ Diverges}
\end{aligned}$$

NOTE: $\lim_{p \rightarrow 0^+} \ln p = -\infty$