Math 121, Section(s) 01, 02, Fall 2025

Homework #17

Due **Wednesday, November 19th** in Gradescope by 11:59 pm ET

Goal: Exploring Estimating Values and Definite Integrals using the Alternating Series Estimation Theorem. Also some review of Interval and Radius of Convergence.

FIRST: Read through and understand the following Examples.

Ex: Use Series to Estimate cos(1) with error less than $\frac{1}{100}$. Justify. Simplify.

First, recall $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$.

Plugging in $x = 1$, we have

$$\cos(1) = \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n}}{(2n)!} \overset{\text{Full Sum}}{=} 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \dots$$

$$\overset{\textit{Estimate}}{\approx} 1 - \frac{1}{2!} + \frac{1}{4!} = 1 - \frac{1}{2} + \frac{1}{24} = \frac{24}{24} - \frac{12}{24} + \frac{1}{24} = \boxed{\frac{13}{24}} \leftarrow \textbf{Estimate}$$

Using the Alternating Series Estimation Theorem (ASET), we can approximate the actual sum with only the first three terms as $\frac{13}{24}$, with error **At Most** the absolute value of the first neglected term, $\frac{1}{6!} = \frac{1}{720} < \frac{1}{100}$ as desired.

Ex: Use Series to Estimate $\int_0^1 x^3 \ln(1 + x^3)\, dx$ with error less than $\frac{1}{30}$. Justify. Simplify.

$$\int_0^1 x^3 \ln(1+x^3)\, dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{n+1}}{n+1}\, dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n+1}\, dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+6}}{n+1}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+7}}{(n+1)(3n+7)}\Bigg|_0^1 = \frac{x^7}{1 \cdot 7} - \frac{x^{10}}{2 \cdot 10} + \frac{x^{13}}{3 \cdot 13} - \dots \Bigg|_0^1$$

$$= \frac{x^7}{7} - \frac{x^{10}}{20} + \frac{x^{13}}{39} - \dots \Bigg|_0^1 = \frac{1}{7} - \frac{1}{20} + \frac{1}{39} - \dots - (0 - 0 + 0 - \dots)$$

$$\approx \frac{1}{7} - \frac{1}{20} = \frac{20}{140} - \frac{7}{140} = \boxed{\frac{13}{140}} \quad \leftarrow \quad \textbf{Estimate}$$

Using the Alternating Series Estimation Theorem (ASET), we can approximate the actual sum with only the first two terms, and the error will be **At Most** the absolute value of the next (first neglected) term, $\frac{1}{39} < \frac{1}{30}$ as desired.

Continue on and Complete the next page of problems

1. Use Series to Estimate $\frac{1}{e}$ with error less than $\frac{1}{20}$. Justify.

2. Use Series to Estimate $\frac{1}{e}$ with error less than $\frac{1}{100}$. Justify. (Can reuse work from 1)

3. Use Series to Estimate $\frac{1}{e}$ with error less than $\frac{1}{500}$. Justify. (Can reuse work from 1)

4. Use Series to Estimate $\sin(1)$ with error less than $\frac{1}{1000}$. Justify.

5. Use Series to Estimate $e^{-\frac{1}{3}}$ with error less than $\frac{1}{100}$. Justify.

6. Use Series to Estimate $\arctan\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$. Justify.

7. Use Series to Estimate $\int_0^1 x \ln(1+x^3)\, dx$ with error less than $\frac{1}{20}$. Justify.

8. Use Series to Estimate $\int_0^1 x \sin(x^2)\, dx$ with error less than $\frac{1}{1000}$. Justify.

Review: Find the Interval and Radius of Convergence for each of the following.

9. $\sum_{n=1}^{\infty} (n!)^2 (3x-7)^n$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n\ (5x-2)^n}{n^3\ 8^n}$

11. $\sum_{n=1}^{\infty} \frac{(x-7)^n}{n!\ \sqrt{n}}$

12. New! Use Series to compute $\lim_{x \to 0} \frac{1-\cos x}{1+x-e^x}$. Check answer with L'H Rule too.

REGULAR OFFICE HOURS

Monday: 12:00–3:00 pm

6–9:00pm TAs Emma/Myles, SMUDD 204

Tuesday: 1:00–4:00 pm

5:30–7:00 pm TA Julia, SMUDD 204

7:30-9:00 pm TA Emma, SMUDD 204

Wednesday: 1:00-3:00 pm

6–10:30 pm TAs Julia/Myles/Natalie, SMUDD 204

Thursday: 10-11:30 am

7:30–10:30 pm TAs Natalie/DJ, SMUDD 204

Friday: 12:00–2:00 pm

7:30–9:00 pm TA DJ, SMUDD 204

Chase the fine details and make a full justification.

YES! Vacation coming!