Math 121, Section(s) 01, 02, Fall 2025

Homework #15

Due MONDAY, November 10th, NOTE CHANGE!!! in Gradescope by 11:59 pm ET

Goal: Exploring Power Series, mainly the Interval and Radius of Convergence. Also beginning to explore the relationship between Power Series and Functions.

FIRST: Read through and understand the following Examples. Determine the Interval and Radius of Convergence. Justify.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(5x-2)^n}{(n+5)\ 8^n}$ Use Ratio Test. $L = \displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(5x-2)^{n+1}}{(n+6)8^{n+1}}}{\frac{\cancel{(-1)^n}(5x-2)^n}{(n+5)8^n}}\right| = \lim_{n\to\infty}\left|\frac{(5x-2)^{n+1}}{(5x-2)^n}\right| \cdot \left(\frac{\cancel{n}+5}{\cancel{n}+6}\right)^{\nearrow 1} \cdot \frac{8^n}{8^{n+1}} = \frac{|5x-2|}{8}$$

The Ratio Test gives convergence for x when $\dfrac{|5x-2|}{8} < 1$ or $|5x-2| < 8$.

That is $-8 < 5x-2 < 8 \Longrightarrow -6 < 5x < 10 \Longrightarrow -\dfrac{6}{5} < x < 2$

Manually Test Endpoints: (where $L = 1$ and Ratio Test is Inconclusive)

•$x = 2$ The original series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(5(2)-2)^n}{(n+5)\ 8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n\cancel{8^n}}{(n+5)\ \cancel{8^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+5}$

which is Convergent by AST: 1. $b_n = \dfrac{1}{n+5} > 0$ 2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n+5} = 0$

3. Terms decreasing: $b_{n+1} = \dfrac{1}{n+6} < \dfrac{1}{n+5} = b_n \Rightarrow$ $x = 2$ is Included in the Domain.

•$x = -\dfrac{6}{5}$ The original series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\left(5\left(-\frac{6}{5}\right)-2\right)^n}{(n+5)\ 8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-8)^n}{(n+5)\ 8^n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n\cancel{8^n}}{(n+5)\cancel{8^n}} = \sum_{n=1}^{\infty} \frac{\cancel{(-1)^{2n}}^{\nearrow 1}}{n+5} = \sum_{n=1}^{\infty} \frac{1}{n+5} \approx \sum_{n=1}^{\infty} \frac{1}{n}$$ the Div Harmonic p-Series $p = 1$.

LCT: $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\cancel{n}+5}^{\nearrow 1} = 1$ which is *Finite* and *Non-zero*. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+5}$ is also Divergent by LCT $\Rightarrow$ $x = -\dfrac{6}{5}$ is NOT included in the Domain.

Finally, Interval of Convergence $\boxed{I = \left(-\frac{6}{5}, 2\right]}$ with Radius of Convergence $\boxed{R = \frac{8}{5}}$.

Ex: $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ Use Ratio Test. $\displaystyle L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty}\left|\frac{\dfrac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\dfrac{x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{x^{2n+1}}\right|\frac{(2n+1)!}{(2n+3)!} = \lim_{n\to\infty}\frac{x^2}{\underbrace{(2n+3)(2n+2)}_{\to\infty}} = 0 < 1 \text{ for all } x$$

Converges by the Ratio Test for all x and $\boxed{I = (-\infty,\infty)}$ with $\boxed{R = \infty}$.

Ex: $\displaystyle\sum_{n=0}^{\infty} n^n (x-7)^n$ Use Ratio Test. $\displaystyle L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}(x-7)^{n+1}}{n^n(x-7)^n}\right| = \lim_{n\to\infty}\underbrace{\frac{(n+1)^n}{n^n}}_{\to e}\underbrace{(n+1)}_{\to\infty}|x-7| = \infty > 1$$

Diverges by the Ratio Test for all x *unless* $x-7=0$ or $x=7$. So $\boxed{I = \{7\}}$ with $\boxed{R = 0}$.

Homework: Determine the Interval and Radius of Convergence for each of the following Power Series. Use the Ratio Test and manually check convergence at the Endpoints for the Finite Intervals. Follow the examples above for statements/format for all three cases. Pay particular attention to the highlighted (necessary) comments above, for each case.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^4\cdot 4^n}$ 3. $\displaystyle\sum_{n=1}^{\infty} n!\ \ln n\ (x-6)^n$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (9x-4)^n}{n^8\cdot 5^n}$ 5. $\displaystyle\sum_{n=0}^{\infty} (3n)!\ (2x-1)^n$ 6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (6x+1)^n}{(6n+1)\cdot 7^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ x^{2n}}{(2n)!}$ 8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (3x-5)^n}{(n+6)^2\cdot 7^{n+1}}$

Find the Power Series Representation for the following functions and determine the Interval of Convergence.

9. $f(x) = \dfrac{1}{1+x}$ 10. $f(x) = \dfrac{5}{1-4x}$ 11. $f(x) = \dfrac{1}{3-x}$

REGULAR OFFICE HOURS

Monday: 12:00–3:00 pm

6–9:00pm TAs Emma/Myles, SMUDD 204

Tuesday: 1:00–4:00 pm

5:30–7:00 pm TA Julia, SMUDD 204

7:30-9:00 pm TA Emma, SMUDD 204

Wednesday: 1:00-3:00 pm

6–10:30 pm TAs Julia/Myles/Natalie, SMUDD 204

Thursday: 10-11:30 am

7:30–10:30 pm TAs Natalie/DJ, SMUDD 204

Friday: 12:00–2:00 pm

7:30–9:00 pm TA DJ, SMUDD 204

Time for a refreshed commitment to the course for a strong finish.