Homework #13

Goal: Exploring Convergence of Infinite Series. Focus on Alternating Series Test, and Ratio Test. We will also focus on fluency of training, using multiple tests.

FIRST: Read through and understand the following Examples. Determine whether the given Series Converges Absolutely or Diverges. Justify.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n! \cdot 2^n}$ Try Ratio Test:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}}{(n+1)!2^{n+1}}}{\frac{n^n}{n!\cdot 2^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}\cdot\frac{n!}{(n+1)!}\cdot\frac{2^n}{2^{n+1}} \quad \text{partner like-terms}$$

$$= \lim_{n\to\infty}\frac{(n+1)^n\cancel{(n+1)}}{n^n}\cdot\frac{\cancel{n!}}{\cancel{(n+1)}\cancel{n!}}\cdot\frac{\cancel{2^n}}{\cancel{2^n}\cdot 2} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n \cdot\left(\frac{1}{2}\right) = \frac{e}{2} > 1$$

The Original Series $\boxed{\text{Diverges by the Ratio Test}}$.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(2n)!}{e^{2n}\cdot n!\cdot n^n}$ Try Ratio Test:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(2(n+1))!}{e^{2(n+1)}\,(n+1)!\,(n+1)^{n+1}}}{\frac{(-1)^n(2n)!}{e^{2n}\,n!\,n^n}}\right|$$

$$= \lim_{n\to\infty}\frac{(2n+2)!}{(2n)!}\cdot\frac{n^n}{(n+1)^{n+1}}\cdot\frac{e^{2n}}{e^{2n+2}}\cdot\frac{n!}{(n+1)!} \quad \text{partner like-terms}$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)(2n)!}{(2n)!}\cdot\frac{n^n}{(n+1)^n(n+1)}\cdot\frac{\cancel{e^{2n}}}{\cancel{e^{2n}}e^2}\cdot\frac{\cancel{n!}}{(n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty}\frac{(2\cancel{(n+1)})(2n+1)}{\cancel{(n+1)}(n+1)}\cdot\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{e^2}$$

$$= \lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\cdot\left(\frac{2}{e^3}\right) = \lim_{n\to\infty}\frac{2+\frac{1}{n}}{1+\frac{1}{n}}\cdot\left(\frac{2}{e^3}\right) = 2\left(\frac{2}{e^3}\right) = \frac{4}{e^3} < 1$$

The original series is $\boxed{\text{Absolutely Convergent (A.C.) by the Ratio Test}}$.

Now complete the following HW problems

1. Consider $\sum_{n=1}^{\infty} \dfrac{n+1}{n^2 + 4n + 7}$. Use **two** Different methods, namely the Integral Test and the Limit Comparison Test, to prove that this series Diverges. *You can skip checking the Integral Test preconditions here this time. yay!*

2. Determine if the given Alternating series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{3n+1}$ Converges or Diverges.

Determine if the given series is Absolutely Convergent or Divergent.

3. $\sum_{n=1}^{\infty} \dfrac{n}{5^n}$

4. $\sum_{n=1}^{\infty} \dfrac{(-3)^n}{(2n+1)!}$

5. $\sum_{n=1}^{\infty} \dfrac{n!}{100^n}$

6. $\sum_{n=1}^{\infty} \dfrac{n!}{n^n}$

7. $\sum_{n=1}^{\infty} \dfrac{n^{100}\,100^n}{n!}$

8. $\sum_{n=1}^{\infty} \dfrac{(2n)!}{(n!)^2}$

$\uparrow$
Recall: $\lim\limits_{n\to\infty} \dfrac{n^n}{(n+1)^n} = \dfrac{1}{e}$

9. Consider the series $\sum_{n=1}^{\infty} \dfrac{\ln n}{n}$.

(a) Show that n^{th} Term Divergence Test is **Inconclusive**.

(b) Show that the Ratio Test is **Inconclusive**.

tip: $\lim\limits_{n\to\infty} \dfrac{\ln(n+1)}{\ln n} \dfrac{\infty}{\infty}$ requires L'Hôpital's Rule.

(c) Show that the series Diverges using the Integral Test. Skip checking the 3 preconditions here. **Note:** This is an example where the terms approach 0 but the series Diverges.

10. Prove that $\sum_{n=1}^{\infty} \dfrac{6}{n^6}$ is Convergent by using the Limit Comparison Test.

Note that this work will be a sample proof of the fact that *Constant multiple of a Convergent series* **is** *Convergent.*

11. Show that $\sum_{n=1}^{\infty} \dfrac{e^n}{n^2}$ Diverges using **two** Different methods.

<u>REGULAR OFFICE HOURS</u>

Monday: 12:00–3:00 pm

6–9:00pm TAs Emma/Myles, SMUDD 204

Tuesday: 1:00–4:00 pm

5:30–7:00 pm TA Julia, SMUDD 204

7:30-9:00 pm TA Emma, SMUDD 204

Wednesday: 1:00-3:00 pm

6–10:30 pm TAs Julia/Myles/Natalie, SMUDD 204

Thursday: 10–11:30 am

extras may be added, TBD weekly

7:30–10:30 pm TAs Natalie/DJ, SMUDD 204

Friday: 12:00–2:00 pm

7:30–9:00 pm TA DJ, SMUDD 204

Train your Convergence Tests Daily