Math 121, Section(s) 01, 02, Fall 2025

**Homework #12**

Due **Wednesday, October 22nd** in Gradescope by 11:59 pm ET

**Goal:** Exploring Convergence of Infinite Series. Focus on Integral Test, $p$-series, Comparison and Limit Comparison Test. We will also focus on fluency of training, using multiple tests.

**FIRST:** Read through and understand the following Examples. Determine whether the given Series Converges or Diverges. Justify with any Convergence Test(s).

Ex: $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$ Related Function $f(x) = \dfrac{\ln x}{x}$ continuous $(x > 0)$, positive $(x > 1)$,

decreasing $f'(x) = \dfrac{1 - \ln x}{x^2} < 0$ for $x > e$. Study the Related Integral

$$\int_2^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_2^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} u\,du = \lim_{t\to\infty} \frac{u^2}{2}\Bigg|_{\ln 2}^{\ln t} = \lim_{t\to\infty} \frac{\cancel{(\ln t)^2}^{\,\infty}}{2} - \frac{(\ln 2)^2}{2} = \boxed{\infty}$$

| $u = \ln x$ <br> $du = \frac{1}{x}\,dx$ | $x = 2 \Rightarrow u = \ln 2$ <br> $x = t \Rightarrow u = \ln t$ |
|---|---|

Therefore, the Improper Integal Diverges. As a result, the Original Series also **Diverges by the Integral Test**. Make sure to have two separate conclusions here. The first implies the second.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + 7} \approx \sum_{n=1}^{\infty} \frac{1}{n^3}$ ← Comparison Series: Convergent $p$-series $p = 3 > 1$

Bound Terms

$\dfrac{1}{n^3 + 7} \le \dfrac{1}{n^3}$. Therefore, the Original Series also **Converges by the Comparison Test.**

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 + 2}{n^4 + 8} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n}$ ← Comparison Series: Divergent $p$-series $p = 1$

Next check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n^3+2}{n^4+8}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^4 + 2n}{n^4 + 8}\frac{\left(\frac{1}{n^4}\right)}{\left(\frac{1}{n^4}\right)} = \lim_{n\to\infty} \frac{1 + \cancel{\frac{2}{n^3}}^{\,0}}{1 + \cancel{\frac{8}{n^4}}^{\,0}} = 1$ Finite and Non-Zero

Therefore, the Original Series also **Diverges by the Limit Comparision Test**.

# Continue to NEXT Page for HW problems.

Use the Integral Test to determine whether the given series Converges or Diverges. You do **NOT** need to check the 3 pre-conditions for the Integral Test this time.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$  2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$  3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$  4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

5. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$. Use **two** Different methods, namely the Integral Test (no pre-Condition check needed) and the Comparison Test, to prove that this series Converges.

Determine if the series Converges or Diverges using either the Comparison **OR** Limit Comparison Test.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$  7. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+5}{n^3}$  8. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$  9. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7}{n^7+2}$

10. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2+n}{n^4}$. Use **two** Different methods to prove that this series Converges. Use the Limit Comparison Test and then a *split-split* algebra technique into $p$-series pieces.

Determine whether the given series Converges or Diverges. Justify.

11. $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{\pi n^4+1}{6n^4+5}\right)$  12. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(\pi n^4+1)}{6n^4+5}$  13. $\displaystyle\sum_{n=1}^{\infty} \frac{7}{n^9} + \frac{7^n}{9^n}$

hint: $\sin(A) \leq 1$
hint: $\sin^2 A \leq 1$ and Comparison Test

**REVIEW**

14. $\displaystyle\sum_{n=1}^{\infty} n^6 + 6$  15. $\displaystyle\sum_{n=1}^{\infty} \frac{n^6+6}{n^6+1}$  16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6+1}$

# REGULAR OFFICE HOURS

## Monday: 12:00–3:00 pm

**6–9:00pm TAs Emma/Myles, SMUDD 204**

## Tuesday: 1:00–4:00 pm

**5:30–7:00 pm TA Julia, SMUDD 204**

**7:30-9:00 pm TA Emma, SMUDD 204**

## Wednesday: 1:00-3:00 pm

**6–10:30 pm TAs Julia/Myles/Natalie, SMUDD 204**

## Thursday: 10–11:30 am

extras may be added, TBD weekly

**7:30–10:30 pm TAs Natalie/DJ, SMUDD 204**

## Friday: 12:00–2:00 pm

**7:30–9:00 pm TA DJ, SMUDD 204**

Train your Convergence Tests Daily