Math 121 Exam 3
December 3, 2025

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{3\ln 3}$, or $\arctan\sqrt{3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [18 Points] Interval/Radius of Convergence Analyze carefully, with full justification.

Find the **Interval** and **Radius** of Convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x+1)^n}{\sqrt{n}\cdot 7^n}$

(b) Design a Power Series centered at $a = 7$ with Radius of Convergence $R = \infty$.

Once you create your series, then proceed to justify that the Interval of Convergence is indeed $I = (-\infty, \infty)$.

Hint: Keep it simple, no need to have complicated coefficient(s).

2. [16 Points] Find the Series for each of the functions. Also **STATE** the Radius of Convergence for each series. Answers should be in sigma notation $\displaystyle\sum_{n=0}^{\infty}$. Simplify.

(a) $\dfrac{d}{dx}\left(5x^4\arctan(5x)\right)$ (b) $\displaystyle\int x^5\cos(6x^2)\,dx$

3. [12 Points] Use Series to **Estimate** $\displaystyle\int_0^1 x^3\sin\left(x^2\right)\,dx$ with error less than $\dfrac{1}{1000}$.

Simplify. Justify. Tip: $(5!)\cdot(14) = 1680$

4. [26 Points] Find the **Sum** value for each of the following convergent series. Simplify.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(36)^n\ (2n+1)!}$ (b) $3 - \frac{3}{2} + 1 - \frac{3}{4} + \frac{3}{5} - \frac{1}{2} + \ldots$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\ \pi^{2n+1}}{4!\ (2n)!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 8)^n}{3^n \cdot n!}$ (e) $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \ldots$ (f) $1 + 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \ldots$

5. [10 Points] Use **Series** to compute the following Limit $\displaystyle\lim_{x\to 0} \frac{5x - \ln(1+5x)}{e^{3x} - 1 - 3x}$

6. [18 Points]

(a) Find the MacLaurin Series Representation for $\ln(9 + x^2)$. **STATE** its Radius of Convergence. Justify.

Hint: Use the formula $\displaystyle\ln(9 + x^2) = \int \frac{2x}{9 + x^2}\, dx \overset{\text{hint}}{=} \int 2x \left(\frac{1}{9 + x^2}\right)\, dx = \ldots.$

(b) Demonstrate **One** method for deriving the MacLaurin Series for $\cos x$. Justify.

You may use the MacLaurin Series formula for $\sin x$ *without extra justification.*

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Create a limit $\displaystyle\lim_{x\to 0} \frac{G(x)}{H(x)}$ which equals $\boxed{-\frac{1}{16}}$ and requires 3 applications of L'Hôpital's Rule. The expression $\frac{G(x)}{H(x)}$ must include at least two of $\sin x$, $\ln(1 + x)$, $\cos x$, $\arctan x$, and e^x. Continue on to compute the Limit using Series, and then also using the three L'H Rules.