Exam 3 Fall 25 Answer Key

1(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(3x+1)^n}{\sqrt{n}\cdot 7^n}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(3x+1)^{n+1}}{\sqrt{n+1}\;\;7^{n+1}}}{\frac{\cancel{(-1)^n}(3x+1)^n}{\sqrt{n}\;\cdot\;7^n}}\right| = \lim_{n\to\infty}\left|\frac{(3x+1)^{n+1}}{(3x+1)^n}\right|\frac{\sqrt{n}}{\sqrt{n+1}}\cdot\frac{\cancel{7^n}}{7^{n+1}} = \lim_{n\to\infty}\frac{|3x+1|}{7}\cancel{\sqrt{\frac{n}{n+1}}}$$

$|3x+1|$

$7^{n+1} = \cancel{7^n}\cdot 7$

$\frac{n}{n+1} \to 1$, $\sqrt{\frac{n}{n+1}} \to 1$

Ratio Test

$$= \lim_{n\to\infty}\frac{|3x+1|}{7} = \frac{|3x+1|}{7} < 1$$

Converges by the Ratio Test when

$$|3x+1| < 7 \Rightarrow \begin{array}{c} -7 < 3x+1 < 7 \\ -1 \qquad\quad -1 \\ -8 < 3x < 6 \end{array} \Rightarrow -\frac{8}{3} < x < 2$$

Manually Test Convergence at Endpoints

Take $x=2$, Series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(3(2)+1)^n}{\sqrt{n}\cdot 7^n} = \sum_{n=0}^{\infty}\frac{(-1)^n \cancel{7^n}}{\sqrt{n}\cdot\cancel{7^n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

Converges by the Alternating Series Test

1. $b_n = \frac{1}{\sqrt{n}} > 0$
2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ ($\frac{1}{\sqrt{n}} \to 0$, $\sqrt{n} \to \infty$)
3. Terms Decreasing
   $b_{n+1} = \frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}} = b_n$

Take $x = -\frac{8}{3}$, Series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n\left(3\left(\frac{-8}{3}\right)+1\right)^n}{\sqrt{n}\cdot 7^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(-7)^n}{\sqrt{n}\;7^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n\cancel{7^n}}{\sqrt{n}\cdot\cancel{7^n}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$$

$(-1)^{2n} \to 1$ even

Divergent

p-Series $p = \frac{1}{2} \le 1$

Finally, $\boxed{I = \left(-\frac{8}{3}, 2\right] \quad R = \frac{7}{3}}$

$2 - \left(-\frac{8}{3}\right) = \frac{6}{3} + \frac{8}{3} = \frac{14}{3} \xrightarrow{\text{half}} \frac{7}{3}$

Total Length

1(b) $\sum_{n=1}^{\infty} \frac{(x-7)^n}{n!}$ OR $\sum_{n=1}^{\infty} \frac{(x-7)^n}{(2n)!}$ OR $\sum_{n=1}^{\infty} \frac{(x-7)^n}{n^n}$ ...

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(x-7)^{n+1}}{(n+1)!}}{\frac{(x-7)^n}{n!}} \right| = \lim_{n\to\infty} \left|\frac{(x-7)^{n+1}}{(x-7)^n}\right| \frac{\cancel{n!}}{(n+1)!}$$

$|x-7|$

$(n+1)\cancel{n!}$

$$= \lim_{n\to\infty} \frac{\overset{\text{Finite}}{|x-7|}}{\underset{\infty}{n+1}} \to 0 = 0 < 1 \quad \text{Converges by Ratio Test for all } x$$

Finally, $\boxed{I = (-\infty, \infty) \quad R = \infty}$

2(a) $\frac{d}{dx}\left(5x^4 \arctan(5x)\right) = \frac{d}{dx}\left(5x^4 \sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{2n+1}}{2n+1}\right)$ Need $|5x|<1$ $\Rightarrow |x| < \frac{1}{5}$ $R = \frac{1}{5}$

$$= \frac{d}{dx}\left(5x^4 \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+1}}{2n+1}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+2} x^{2n+5}}{2n+1}\right)$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+2} (2n+5) x^{2n+4}}{2n+1}}$$

$R = \frac{1}{5}$ STILL After Differentiation

2(b) $\int x^5 \cos(6x^2)\,dx = \int x^5 \sum_{n=0}^{\infty} \frac{(-1)^n (6x^2)^{2n}}{(2n)!}\,dx = \int x^5 \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n} x^{4n}}{(2n)!}\,dx$

$R = \infty$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n} x^{4n+5}}{(2n)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n} x^{4n+6}}{(2n)!\,(4n+6)} + C$$

$R = \infty$ STILL

After Integration

$$3 \quad \int_0^1 x^3 \sin(x^2)\,dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}\,dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\,dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+6}}{(2n+1)!(4n+6)} \Bigg|_0^1$$

$n=0 \quad n=1 \quad n=2 \quad \ldots$

$$= \frac{x^6}{1\cdot 6} - \frac{x^{10}}{3!\cdot(10)} + \frac{x^{14}}{\underbrace{5!\,(14)}_{1680}} - \ldots \Bigg|_0^1$$

Tip

Full Sum

Show Zeroes

$$= \frac{1}{6} - \frac{1}{60} + \frac{1}{1680} - \ldots - \cancel{(0-0+0-\ldots)}^{\,0}$$

$$\approx \frac{1}{6} - \frac{1}{60} = \frac{10}{60} - \frac{1}{60} = \frac{9}{60} = \frac{3}{20} \quad \leftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem (A.S.E.T.) we can Estimate the full Sum using only the first two terms with Error at most $\frac{1}{1680} < \frac{1}{1000}$ as desired

$$4(a) \quad \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{\underbrace{(36)^n}_{6^{2n}}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n+1)!} \cdot \frac{\frac{\pi}{6}}{\frac{\pi}{6}} = \frac{1}{\frac{\pi}{6}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n+1}}{(2n+1)!}$$

extra

$$= \frac{6}{\pi} \cancel{\sin\left(\frac{\pi}{6}\right)}^{\,\frac{1}{2}} = \frac{6}{\pi} \cdot \frac{1}{2} = \frac{3}{\pi}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4(b) $3-\frac{3}{2}+1-\frac{3}{4}+\frac{3}{5}-\frac{1}{2}+\ldots = 3-\frac{3}{2}+\frac{3}{3}-\frac{3}{4}+\frac{3}{5}-\frac{3}{6}+\ldots$

$= 3\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\right)$

$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$

$= 3\ln(1+1) = 3\ln 2$

4(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n+1}}{4!\,(2n)!} = -\frac{\pi}{4!}\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!} = -\frac{\pi}{24}\cancelto{-1}{\cos\pi} = \frac{\pi}{24}$

$\cos x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

4(d) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(\ln 8)^n}{3^n\cdot n!} = \sum_{n=0}^{\infty}\frac{\left(\frac{-\ln 8}{3}\right)^n}{n!} = e^{\frac{-\ln 8}{3}} = e^{-\frac{1}{3}\ln 8}$

$e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$= \cancel{e}^{\ln\left(8^{-1/3}\right)} = 8^{-\frac{1}{3}} = \frac{1}{8^{1/3}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$

4(e) $-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\ldots = -\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\right)$

$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$

$= -\cancelto{\frac{\pi}{4}}{\arctan 1} = -\frac{\pi}{4}$

Note: $\arctan(-1) = -\frac{\pi}{4}$ also works here

4(f) $1+1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\frac{\pi^8}{8!}-\ldots = 1+\cancelto{-1}{\cos\pi} = 0$

$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

5. $$\lim_{x\to 0}\frac{5x-\ln(1+5x)}{e^{3x}-1-3x}=\lim_{x\to 0}\frac{\cancel{5x}-\left(\cancel{5x}-\frac{(5x)^2}{2}+\frac{(5x)^3}{3}-\frac{(5x)^4}{4}+\dots\right)}{\left(\cancel{1}+\cancel{3x}+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\dots\right)-\cancel{1}-\cancel{3x}}$$

$$=\lim_{x\to 0}\frac{\frac{25}{2}x^2-\frac{5^3x^3}{3}+\frac{5^4x^4}{4}-\dots}{\frac{9x^2}{2!}+\frac{3^3x^3}{3!}+\frac{3^4x^4}{4!}+\dots}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to 0}\frac{\frac{25}{2}-\frac{5^3x}{3}+\frac{5^4x^2}{4}-\dots}{\frac{9}{2}+\frac{3^3x}{3!}+\frac{3^4x^2}{4!}+\dots}=\frac{\frac{25}{\cancel{2}}}{\frac{9}{\cancel{2}}}=\frac{25}{9}$$

Optional: Can check answer using TWO L'H Rules

$-5(1+5x)^{-1} \xrightarrow{\frac{d}{dx}} +5(1+5x)^{-2}\cdot 5$

$$\lim_{x\to 0}\frac{5x-\ln(1+5x)}{e^{3x}-1-3x}\ \frac{0}{0}\ \underset{L'H}{=}\ \lim_{x\to 0}\frac{5-\frac{5}{1+5x}}{3e^{3x}-3}\ \frac{0}{0}\ \underset{L'H}{=}\ \lim_{x\to 0}\frac{\frac{25}{(1+5x)^2}}{9e^{3x}}=\frac{25}{9}$$ Match!

6(a) $$\ln(9+x^2)=\int\frac{2x}{9+x^2}dx=\int 2x\left(\frac{1}{9+x^2}\right)dx=\int\frac{2x}{9}\left(\frac{1}{1+\frac{x^2}{9}}\right)dx=\int\frac{2x}{9}\left(\frac{1}{1-\left(-\frac{x^2}{9}\right)}\right)dx$$

$$=\int\frac{2x}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{9}\right)^n dx=\int\frac{2x}{9}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{9^n}dx=\int 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{9^{n+1}}dx$$

$$=2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{9^{n+1}(2n+2)}+C=2\left(\underset{n=0}{\frac{x^2}{9\cdot 2}}-\underset{n=1}{\frac{x^4}{9^2\cdot 4}}+\underset{n=2}{\frac{x^6}{9^3\cdot 6}}-\dots\right)+C$$

Optional (this time) Solve for +C

Need $\left|-\frac{x^2}{9}\right|<1 \Rightarrow |x|^2<9$

$\Rightarrow |x|<3$

$R=3$

Test the center $x=0$ to solve for $+C$

$$\ln(9+0)=2(0-0+0-\dots)+C \Rightarrow C=\ln 9$$

$\Rightarrow R=3$ After Integration

Finally, $$\ln(9+x^2)=2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{9^{n+1}(2n+2)}+\ln 9$$

6(b) present either "Chart Method" (by Definition) OR Differentiation OR Integration

Sample: $\cos x = \frac{d}{dx}\left(\sin x\right) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\cancel{(2n+1)}x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

$(2n+1)! = \cancel{(2n+1)}(2n)!$

Optional Bonus: Create $\lim_{x\to 0}\frac{G(x)}{H(x)} = -\frac{1}{16}$

$2(1+4x^2)^{-1} \to -2(1+4x^2)^{-2}(8x)$

$$\lim_{x\to 0}\frac{x-\sin x}{\arctan(2x)-2x}\ \frac{0}{0} \underset{L'H}{=} \lim_{x\to 0}\frac{1-\cos x}{\frac{2}{1+(2x)^2}-2}\ \frac{0}{0} \underset{L'H}{=} \lim_{x\to 0}\frac{+\sin x}{\frac{-16x}{(1+4x^2)^2}}\ \frac{0}{0}$$

Quotient Rule

$$\underset{L'H}{=} \lim_{x\to 0}\frac{\cancel{\cos x}\to 1}{\frac{\cancel{(1+4x^2)^2}(-16)-\cancel{(-16x)2(1+4x^2)(8x)}\to 0}{(1+4x^2)^4\to 1}} = -\frac{1}{16}$$

yuck!

$$\lim_{x\to 0}\frac{x-\sin x}{\arctan(2x)-2x} = \lim_{x\to 0}\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)}{2x-\frac{(2x)^3}{3}+\frac{(2x)^5}{5}-\dots-2x}$$

$$= \lim_{x\to 0}\frac{\cancel{x}-\cancel{x}+\frac{x^3}{3!}-\frac{x^5}{5!}+\dots}{\cancel{2x}-\frac{2^3x^3}{3}+\frac{2^5x^5}{5}-\dots-\cancel{2x}}$$

$$= \lim_{x\to 0}\frac{\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\dots}{-\frac{2^3x^3}{3}+\frac{2^5x^5}{5}-\frac{2^7x^7}{7}+\dots}\ \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to 0}\frac{\frac{1}{3!}-\frac{x^2}{5!}+\frac{x^4}{7!}-\dots \to 0}{-\frac{8}{3}+\frac{2^5x^2}{5}-\frac{2^7x^4}{7}+\dots \to 0} = \frac{\frac{1}{3!}}{\frac{-8}{3}}\ \frac{-3}{8} = \frac{-1}{2\cdot 8} = \frac{-1}{16}$$

Yes! Match!

OR //

$$\lim_{x\to 0} \frac{x\cos x - x}{48\left(e^x - 1 - x - \frac{x^2}{2!}\right)}$$

helps scale

$$\cancel{x} - \frac{x^3}{2!} + \frac{x^4}{4!} - \dots \cancel{-x}$$

$$= \lim_{x\to 0} \frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - x}{48\left(\cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \cancel{-1} \cancel{-x} \cancel{-\frac{x^2}{2!}}\right)}$$

$$= \lim_{x\to 0} \frac{-\frac{x^3}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}{48\left(\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right)} \quad \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{2!} + \cancel{\frac{x}{4!} - \frac{x^3}{6!} + \dots}^{\to 0}}{48\left(\frac{1}{3!} + \cancel{\frac{x}{4!} + \frac{x^2}{5!} + \dots}^{\to 0}\right)} = \frac{-\frac{1}{2}}{48\left(\frac{1}{6}\right)} = \frac{-\frac{1}{2}}{\underset{1}{8}} \, \frac{1}{8} = -\frac{1}{16}$$

$$\lim_{x\to 0} \frac{x\cos x - x}{48\left(e^x - 1 - x - \frac{x^2}{2!}\right)} \quad \frac{0}{0} \quad \underset{\text{L'H}}{=} \lim_{x\to 0} \frac{x(-\sin x) + \cos x\,(1) - 1}{48\left(e^x - 1 - x\right)} \quad \frac{0}{0}$$

$- 2\sin x$

$$\underset{\text{L'H}}{=} \lim_{x\to 0} \frac{x(-\cos x) + (-\sin x)\cdot 1 - \sin x}{48\left(e^x - 1\right)} \quad \frac{0}{0}$$

$$\underset{\text{L'H}}{=} \lim_{x\to 0} \frac{x(\sin x)^{\to 0} + (-\cos x)(1)^{\to -1} - 2\cos x^{\to -1}}{48\, e^{x \to 1}}$$

$$= \frac{-3}{48} = -\frac{1}{16}$$

Yes! Match!

OR

$$\lim_{x\to 0}\frac{\ln(1+x)-x+\frac{x^2}{2}}{4(\sin(2x)-2x)} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{\frac{1}{1+x}-1+x}{4(2\cos(2x)-2)} \quad \frac{0}{0}$$

helps scale

$$\underset{\text{L'H}}{=} \lim_{x\to 0}\frac{\frac{-1}{(1+x)^2}+1}{-16\sin(2x)} \quad \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{\frac{2}{(1+x)^3} \to 2}{-32\cos(2x) \to 1}$$

$$= \frac{2}{-32} = -\frac{1}{16}$$

$$\lim_{x\to 0}\frac{\ln(1+x)-x+\frac{x^2}{2}}{4(\sin(2x)-2x)} = \lim_{x\to 0}\frac{\cancel{x}-\cancel{\frac{x^2}{2}}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\cancel{-x}\cancel{+\frac{x^2}{2}}}{4\left(\cancel{2x}-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\dots\cancel{-2x}\right)}$$

$$= \lim_{x\to 0}\frac{\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\dots}{4\left(-\frac{8x^3}{3!}+\frac{2^5x^5}{5!}-\frac{2^7x^7}{7!}+\dots\right)} \quad \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to 0}\frac{\frac{1}{3}-\frac{x}{4}+\frac{x^2}{5}-\dots \to 0}{4\left(\frac{-8}{3!}+\frac{2^5x^2}{5!}-\frac{2^7x^4}{7!}+\dots\right) \to 0}$$

$$= \frac{\frac{1}{3}}{\frac{-32}{6}} = \frac{1}{\cancel{3}}\cdot\frac{\cancel{6}^{\,2}}{-32} = \frac{-2}{32} = -\frac{1}{16}$$

Yes! Match!