Exam 2 Fall 25 Answer Key

1(a) $\displaystyle\int_{0}^{e} x^4 \cdot \ln x\,dx = \lim_{t\to 0^+}\int_t^e x^4 \cdot \ln x\,dx = \lim_{t\to 0^+} \frac{x^5}{5}\cdot \ln x\Big|_t^e - \frac{1}{5}\int_t^e x^4\,dx$

$$\boxed{\begin{array}{ll} u=\ln x & dv = x^4\,dx \\ du=\frac{1}{x}\,dx & v=\frac{x^5}{5}\end{array}}$$

$$= \lim_{t\to 0^+} \frac{x^5}{5}\cdot \ln x\Big|_t^e - \frac{x^5}{25}\Big|_t^e$$

$$= \lim_{t\to 0^+} \frac{e^5}{5}\cdot \cancelto{1}{\ln e} - \cancelto{0}{\overbrace{\frac{t^5}{5}\ln t}^{0\cdot(-\infty)}} - \left(\frac{e^5}{25} - \cancelto{0}{\frac{t^5}{25}}\right)$$

☆ see below

$$= \frac{e^5}{5} - \frac{e^5}{25} = \frac{5e^5}{25} - \frac{e^5}{25} = \frac{4e^5}{25} \quad \text{Converges}$$

$$☆ \lim_{t\to 0^+} \overbrace{t^5\cdot \ln t}^{0\cdot(-\infty)} = \lim_{t\to 0^+} \frac{\ln t}{1/t^5}\;\left(\frac{-\infty}{\infty}\right) \underset{\text{L'H}}{=} \lim_{t\to 0^+}\frac{\frac{1}{t}}{\frac{-5}{t^6}} = \lim_{t\to 0^+} \frac{\cancelto{0}{-t^5}}{5} = 0$$

$t^{-5} \to -5t^{-6}$ $\qquad \frac{t^6}{-5}$

1(b) $\displaystyle\int_{-\infty}^{1} \frac{1}{x^2-6x+13}\,dx = \lim_{t\to -\infty}\int_t^1 \frac{1}{x^2-6x+13}\,dx = \lim_{t\to-\infty}\int_t^1 \frac{1}{(x-3)^2+4}\,dx$

Complete the Square $\qquad x^2-6x+9+\underline{4}$

$$\boxed{\begin{array}{l} u=x-3 \\ du=dx\end{array}}$$

$$\boxed{\begin{array}{l} x=t \Rightarrow u=x-3 \\ x=1 \Rightarrow u=-2\end{array}}$$

$$= \lim_{t\to-\infty}\int_{t-3}^{-2}\frac{1}{u^2+4}\,du = \lim_{t\to-\infty}\frac{1}{2}\arctan\left(\frac{u}{2}\right)\Big|_{t-3}^{-2}$$

$$= \lim_{t\to-\infty}\frac{1}{2}\left(\cancelto{-\frac{\pi}{4}}{\arctan\left(\cancelto{-1}{\frac{-2}{2}}\right)} - \cancelto{-\frac{\pi}{2}}{\arctan\left(\cancelto{-\infty}{\frac{t-3}{2}}\right)}\right)$$

$$= \frac{1}{2}\left(-\frac{\pi}{4} \overset{--}{+} \underbrace{\frac{\pi}{2}}_{\frac{2\pi}{4}}\right) = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8} \quad \text{Converges}$$

$$1(c)\quad \int_{-7}^{0} \frac{x+15}{x^2+6x-7}\,dx = \lim_{t\to -7^+} \int_t^0 \frac{x+15}{x^2+6x-7}\,dx = \lim_{t\to -7^+} \int_t^0 \frac{x+15}{(x-1)(x+7)}\,dx$$

PFD Given for free this Semester

$$\cancel{(x+7)(x-1)}\left[\frac{x+15}{\cancel{(x-1)(x+7)}} = \frac{A}{x-1} + \frac{B}{x+7}\right](x+7)(x-1)$$

$$x+15 = A(x+7) + B(x-1)$$

$$= Ax + 7A + Bx - B$$

$$= (A+B)x + (7A-B)$$

Conditions

- $A+B=1 \Rightarrow B = 1-A$
- $7A - B = 15$

$$7A-(1-A)=15$$

$$7A-1+A=15$$

$$8A=16$$

$$A=2 \longrightarrow B=1-2=-1$$

PFD

$$= \lim_{t\to -7^+} \int_t^0 \frac{2}{x-1} - \frac{1}{x+7}\,dx$$

$$= \lim_{t\to -7^+} 2\ln|x-1| - \ln|x+7| \Big|_t^0$$

$$= \lim_{t\to -7^+} 2\cancel{\ln|-1|}^{\,0} - \ln 7 - \left(\underbrace{2\ln|t-1|}_{\text{Finite}} - \ln|\cancel{t+7}|\right)$$

Finite, $2\ln 8$, $-\infty$, -7^+, 0^+

$$= -(-(-\infty)) = -\infty \qquad \text{Diverges}$$

$$2.\quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^4} \xrightarrow{\text{A.S.}} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4} \quad \text{Integral Test}$$

Related Function $f(x) = \dfrac{1}{x(\ln x)^4}$

Not Equal

$$\int_2^{\infty} \frac{1}{x(\ln x)^4}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^4}\,dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^4}\,du$$

$$u^{-4} \nearrow \frac{u^{-3}}{-3}$$

u-sub

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}}$$

$$\boxed{\begin{aligned} x=2 &\Rightarrow u=\ln 2 \\ x=t &\Rightarrow u=\ln t \end{aligned}}$$

$$= \lim_{t\to\infty} \frac{-1}{3u^3}\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} -\cancel{\frac{1}{3(\ln t)^3}}^{\,0} + \frac{1}{3(\ln 2)^3}$$

$$= \underbrace{\frac{1}{3(\ln 2)^3}}_{\text{Finite}} \quad \text{Integral Converges}$$

$\Rightarrow$ Absolute Series Converges by the Integral Test

Important

Two Separate Conclusions

ACT

Original Series Converges by the Absolute Convergence Test

3. $\displaystyle\sum_{n=2}^{\infty} \frac{n^6}{\ln n}$ $\rightarrow$ $\boxed{\text{Diverges by nTDT}}$ because

$$\lim_{n\to\infty} \frac{n^6}{\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{x^6}{\ln x} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=}} \lim_{x\to\infty} \frac{6x^5}{\frac{1}{x}} = \lim_{x\to\infty} 6x^6 = \infty \neq 0$$

OR// Alternate Examples

$$\sum_{n=1}^{\infty} \frac{e^n}{n^2} \qquad \sum_{n=2}^{\infty} \frac{e^{3n}}{\ln n} \qquad \sum_{n=1}^{\infty} \left(1+\frac{5}{n^2}\right)^{n^2} \qquad \ldots$$

4(a) $\displaystyle\sum_{n=1}^{\infty} (2025)! = (2025)! + (2025)! + (2025)! + \ldots$

$r=1$

Diverges by Geometric Series Test with $|r| = 1 \leq 1$

OR// Diverges by n^{th} Term Divergence Test because

$$\lim_{n\to\infty} (2025)! = (2025)! \neq 0$$

Some very Large Constant

4(b) $-4 - \frac{4}{2} - \frac{4}{3} - 1 - \frac{4}{5} - \frac{4}{6} - \ldots = -4\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots\right)$

$-\frac{4}{4}$

$$= -4 \sum_{n=1}^{\infty} \frac{1}{n}$$

Constant Multiple of the Divergent (Harmonic) p-Series $p=1$ is $\boxed{\text{Divergent}}$

4(c) $\sum_{n=1}^{\infty} \frac{4}{n^6} + \frac{(-6)^n}{7^{2n}}$ split $= \sum_{n=1}^{\infty} \frac{4}{n^6} + \sum_{n=1}^{\infty} \frac{(-6)^n}{7^{2n}}$

$r = \frac{-6}{49}$

$= 4\sum_{n=1}^{\infty} \frac{1}{n^6} + \sum_{n=1}^{\infty} \frac{(-6)^n}{49^n}$ $\qquad \frac{-6}{49} + \frac{6^2}{(49)^2} - \frac{6^3}{(49)^3} + \dots$

Constant Multiple of a Convergent p-Series $p = 6 > 1$ is Convergent

Convergent by GST with $|r| = \left|\frac{-6}{49}\right| = \frac{6}{49} < 1$

Sum of Two Convergent Series is $\boxed{\text{Convergent}}$

5(a) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^4+6}{n^6+4}\right) \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty} \frac{n^4+6}{n^6+4} \approx \sum_{n=1}^{\infty} \frac{n^4}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ Convergent p-Series $p = 2 > 1$

don't need to study the original series

LCT Limit

$\lim_{n\to\infty} \frac{\frac{n^4+6}{n^6+4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^6+6n^2}{n^6+4} \cdot \frac{\frac{1}{n^6}}{\frac{1}{n^6}} = \lim_{n\to\infty} \frac{1+\frac{6}{n^4}}{1+\frac{4}{n^6}} = 1$ Finite Non-Zero

$\Rightarrow$ the Absolute Series also Converges by LCT

$\Rightarrow$ the Original Series is $\boxed{\text{Absolutely Convergent}}$ (by Definition)

5 (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^n (2n)!}{6^n (n!)^3}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}} (n+1)^{n+1} (2(n+1))!}{6^{n+1} ((n+1)!)^3}}{\dfrac{\cancel{(-1)^n} n^n (2n)!}{6^n (n!)^3}} \right|$$

$\dfrac{6^n (n!)^3}{n^n (2n)!}$

$(n+1)^n (n+1) \qquad (2n+2)(2n+1)\cancel{(2n)!}$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{(2n+2)!}{\cancel{(2n)!}} \cdot \frac{\cancel{6^n}}{6^{n+1}} \cdot \frac{\cancel{(n!)^3}}{((n+1)!)^3}$$

$\cancel{6^n} \cdot 6$

$((n+1)n!)^3$

$(n+1)^3 \cancel{(n!)^3}$

3 copies

$$= \lim_{n\to\infty} \underbrace{\frac{(n+1)^n}{n^n}}_{e} \cdot \frac{1}{6} \cdot \underbrace{\left(\frac{n+1}{n+1}\right)}_{1} \left(\frac{\overbrace{2n+2}^{2(n+1)}}{\cancel{n+1}}\right) \left(\frac{2n+1}{n+1} \frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$$= \lim_{n\to\infty} \frac{2e}{6} \left(\frac{2 + \frac{1}{n} \to 0}{1 + \frac{1}{n} \to 0}\right) = \frac{4e}{6} = \frac{2e}{3} > 1$$

$\boxed{\text{Original Series Diverges by Ratio Test}}$

$2e > 4$

$e \approx 2.7$

5(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{6n+4} \xrightarrow[\text{A.S.}]{\text{1st}} \sum_{n=1}^{\infty} \frac{1}{6n+4} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Divergent (Harmonic) p-Series $p=1$

2nd AST

1. Isolate $b_n = \dfrac{1}{6n+4} > 0$

2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{6n+4} = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{\underbrace{6(n+1)+4}_{6n+10}} \leq \frac{1}{6n+4} = b_n$$

Original Series Converges by AST

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{1}{6n+4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{6n+4} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{6+\frac{4}{n}} = \frac{1}{6}$$

Finite + Non-Zero

$\Rightarrow$ Absolute Series Diverges by LCT

Original Series is $\boxed{\text{Conditionally Convergent}}$ by Definition

Optional Bonus:

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$$

First: Terms approach 0 as $n\to\infty$ $\quad \displaystyle\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = 0$ ✓

Second: Use the Integral Test to show the Series Diverges

$$\int_1^{\infty} \ln\left(1+\frac{1}{x}\right) dx = \lim_{t\to\infty} \int_1^t \ln\left(1+\frac{1}{x}\right)\cdot 1 \, dx$$

L · P

IBP

$$\boxed{\begin{array}{ll} u = \ln\left(1+\frac{1}{x}\right) & dv = 1\,dx \\ du = \frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)dx & v = x \end{array}}$$

$$-\frac{1}{x^2+x}$$

$$= \lim_{t\to\infty} x\cdot \ln\left(1+\frac{1}{x}\right)\Big|_1^t \; \overset{--}{+} \int_1^t \frac{1}{x+1}\,dx$$

Bonus (Continued)

$$= \lim_{t\to\infty} x\cdot\ln\left(1+\frac{1}{x}\right)\Big|_1^t + \ln|x+1|\Big|_1^t$$

$$= \lim_{t\to\infty} \underbrace{\overset{\infty\cdot 0}{t\cdot\ln\left(1+\frac{1}{t}\right)}}_{\text{Finite}} - \underbrace{1\cdot\ln 2}_{\text{Finite}} + \ln|\overset{\infty}{\cancel{t+1}}| - \underbrace{\ln 2}_{\text{Finite}}$$

$= \infty$ Integral Diverges, therefore the $\boxed{\text{Series Diverges by Integral Test}}$

$$\lim_{t\to\infty} t\cdot\ln\left(1+\frac{1}{t}\right) = \lim_{t\to\infty}\frac{\ln\left(1+\frac{1}{t}\right)}{\frac{1}{t}} \overset{0/0}{\underset{L'H}{=}} \lim_{t\to\infty}\frac{\frac{1}{1+1/t}\left(\cancel{-\frac{1}{t^2}}\right)}{\cancel{-\frac{1}{t^2}}} = \lim_{t\to\infty}\frac{1}{1+\frac{1}{t}} = 1$$

Alternate ?

Note: Can also use Algebra

$$\int_1^\infty \ln\left(1+\frac{1}{x}\right)dx = \int_1^\infty \ln\left(\frac{x+1}{x}\right)dx = \int_1^\infty \ln|x+1| - \ln x \, dx$$

and run two separate IBPs

but may require more work

on the Indeterminate Product

Limit Finishes