1. Find the **sum** of the following series $\sum_{n=1}^{\infty}(-1)^n \frac{6^{n+1}}{5^{3n-1}}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, 6^{n+1}}{5^{3n-1}} = -\frac{6^2}{5^2} + \frac{6^3}{5^5} - \frac{6^4}{5^8} + \dots$$

Here we have a Geometric series with $a = -\frac{36}{25}$ and $r = -\frac{6}{5^3} = -\frac{6}{125}$.

Note, it does converge since $|r| = \left|-\frac{6}{125}\right| = \frac{6}{125} < 1$.

As a result, the sum is given by SUM$=\frac{a}{1-r} = \frac{-\frac{36}{25}}{1-\left(-\frac{6}{125}\right)} = \frac{-\frac{36}{25}}{\frac{131}{125}} = -\frac{36}{25}\cdot\frac{125}{131} = \boxed{-\frac{180}{131}}$

2. Use the **Integral Test** to **determine** and **state** whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ Converges or Diverges. Justify all of your work. You can skip the 3 preconditions.

Check the improper integral

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{t\to\infty} \int_1^t (\ln x) x^{-2}\, dx \overset{\text{IBP}}{=} \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_1^t + \int_1^t x^{-2}\, dx$$

$$= \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_1^t - \frac{1}{x}\Big|_1^t = \lim_{t\to\infty} -\frac{\ln t}{t}^{\frac{\infty}{\infty}} + \frac{\cancel{\ln 1}^{\,0}}{1} - \left(\cancel{\frac{1}{t}}^{\,0} - \frac{1}{1}\right)$$

$$\overset{\text{L'H}}{=} \lim_{t\to\infty} -\frac{\left(\cancel{\frac{1}{t}}^{\,0}\right)}{1} + 0 - 0 + 1 = 1$$

The improper integral Converges, and therefore the original series $\boxed{\text{Converges}}$ by the Integral Test.

IBP:

$u = \ln x$	$dv = x^{-2}dx$
$du = \frac{1}{x}dx$	$v = -\frac{1}{x}$

Note: it was not required to check the 3 pre-conditions here, but if you did, they would be as follows:

Consider the related function $f(x) = \dfrac{\ln x}{x^2}$ with

1. $f(x)$ continuous for all $x > 0$
2. $f(x)$ positive for $x > 1$
3. $f(x)$ decreasing because $f'(x) = \dfrac{x^2\left(\frac{1}{x}\right) - \ln x(2x)}{\left(x^2\right)^2} = \dfrac{1 - 2\ln x}{x^3} < 0$ when $x > e^{\frac{1}{2}}$.

3. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 7}$

Use two Different methods, namely the Integral Test (no pre-Condition check needed) and the Comparison Test, to prove that this series Converges.

First, the Integral test:

Check the improper integral

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2 + 4x + 7}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2 + 4x + 7}\,dx \\
&= \lim_{t\to\infty} \int_1^t \frac{1}{(x+2)^2 + 3}\,dx \quad \text{Complete the Square} \\
&= \lim_{t\to\infty} \int_3^{t+2} \frac{1}{u^2 + 3}\,du \\
&= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_3^{t+2} \\
&= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \left[\cancel{\arctan}\,(\to \tfrac{\pi}{2}) \left(\cancel{\frac{t+2}{\sqrt{3}}} \to \infty\right) - \arctan\left(\cancel{\frac{3}{\sqrt{3}}} \to \sqrt{3}\right) \right] \\
&= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{3\pi}{6} - \frac{2\pi}{6}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \frac{\pi}{6\sqrt{3}}
\end{aligned}$$

$u = x + 2$	$x = 1 \Rightarrow u = 3$
$du = dx$	$x = t \Rightarrow u = t + 2$

The improper integral Converges, and therefore the original series

Converges by the Integral Test.

Second, the Comparison Test:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+7} \approx \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ which is a Convergent } p\text{-series with } p = 2 > 1$$

Bound Terms: $\dfrac{1}{n^2+4n+7} < \dfrac{1}{n^2}$

Therefore, the Original Series $\boxed{\text{Converges by the Comparison Test}}$

Note: The Limit Comparison Test also works here as a third option.

In each case determine whether the given series **Converges**, or **Diverges**. Name any Convergence Test(s) you use, and justify all of your work.

4. $\sum_{n=1}^{\infty} n^6 + 7$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$ because $\lim_{n\to\infty} \cancelto{\infty}{n^6} + 7 = \infty \neq 0$

5. $\sum_{n=1}^{\infty} \dfrac{n^6+7}{n^6+1}$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$ because

$$\lim_{n\to\infty} \frac{n^6+7}{n^6+1} = \lim_{n\to\infty} \frac{n^6+7}{n^6+1} \cdot \frac{\frac{1}{n^6}}{\frac{1}{n^6}} = \lim_{n\to\infty} \frac{1+\cancelto{0}{\frac{7}{n^6}}}{1+\cancelto{0}{\frac{1}{n^6}}} = 1 \neq 0$$

6. $\sum_{n=1}^{\infty} \dfrac{1}{n^6+1} \approx \sum_{n=1}^{\infty} \dfrac{1}{n^6}$ which is a Convergent p-series with $p = 6 > 1$.

Bound Terms: $\dfrac{1}{n^6+1} < \dfrac{1}{n^6}$

Therefore, the Original Series $\boxed{\text{Converges by the Comparison Test}}$

Note: Limit Comparison Test will also work here.

7. $\sum_{n=1}^{\infty} \dfrac{n^6+7}{n^7+1} \approx \sum_{n=1}^{\infty} \dfrac{n^6}{n^7} = \sum_{n=1}^{\infty} \dfrac{1}{n}$ which is the Divergent Harmonic p-series with $p = 1$

Study the Comparison Limit:

$$\lim_{n\to\infty} \frac{\dfrac{n^6+7}{n^7+1}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n^7+7n}{n^7+1}\cdot\frac{\frac{1}{n^7}}{\frac{1}{n^7}} = \lim_{n\to\infty} \frac{1+\cancelto{0}{\frac{7}{n^6}}}{1+\cancelto{0}{\frac{1}{n^7}}} = 1 \text{ which is Finite and Non-Zero.}$$

They share the *same behavior.*

Therefore, the Original Series is also $\boxed{\text{Divergent by the Limit Comparison Test}}$.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n+6}{n^7+1} \approx \sum_{n=1}^{\infty} \frac{n}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^6}$ which is a Convergent p-series with $p = 6 > 1$

Study the Comparison Limit:

$$\lim_{n\to\infty} \frac{\dfrac{n+6}{n^7+1}}{\dfrac{1}{n^6}} = \lim_{n\to\infty} \frac{n^7+6n^6}{n^7+1}\cdot\frac{\frac{1}{n^7}}{\frac{1}{n^7}} = \lim_{n\to\infty} \frac{1+\cancelto{0}{\frac{6}{n}}}{1+\cancelto{0}{\frac{1}{n^7}}} = 1 \text{ which is Finite and Non-Zero.}$$

They share the *same behavior.*

Therefore, the Original Series is also $\boxed{\text{Convergent by the Limit Comparison Test}}$.

Spend some time comparing and contrasting problems 8 and 9. They are structurally very similar Limit Comparison Tests, with one converging and one diverging. Remember, that if the Comparison stacked limit is Finite and Non-Zero, it does not automatically mean *Converge*. Instead, it means the Original Series *shares the same convergence behavior* as the Comparison Series. *They do the same thing!* That is, the Original and Comparison Series either both Converge or both Diverge.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^6} + \frac{5^n}{6^n} \overset{\text{split}}{=} \sum_{n=1}^{\infty} \frac{5}{n^6} + \sum_{n=1}^{\infty} \frac{5^n}{6^n} = 5\sum_{n=1}^{\infty} \frac{1}{n^6} + \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n$

The first series is Convergent because a Constant Multiple of a Convergent p-Series with $p = 6 > 1$ is convergent.

The second series is Convergent Geometric Series with $|r| = \left|\dfrac{5}{6}\right| = \dfrac{5}{6} < 1$

Finally, the Original Series is $\boxed{\text{Convergent as the Sum of Two Convergent series}}$.

10. $\displaystyle\sum_{n=2}^{\infty} \frac{n^6}{\ln n}$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$ because

$$\lim_{n\to\infty} \frac{n^6}{\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{x^6}{\ln x} \overset{\frac{\infty}{\infty}}{\underset{}{\overset{\text{L'H}}{=}}} \lim_{x\to\infty} \frac{6x^5}{\frac{1}{x}} = \lim_{x\to\infty} 6x^6 \to \infty = \infty \neq 0$$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln 6}{n^6} = \ln 6 \sum_{n=1}^{\infty} \frac{1}{n^6}$

is $\boxed{\text{Convergent}}$ because a constant multiple of a Convergent p-series with $p = 6 > 1$ is Convergent.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6^{2n}} = \sum_{n=1}^{\infty} \left(\frac{1}{6^2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{36}\right)^n$

is a $\boxed{\text{Convergent Geometric series}}$ with $|r| = \left|\frac{1}{36}\right| = \frac{1}{36} < 1$.

Note: once we gain the Ratio Test in class, you can use the Ratio test here too.

13. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{6}{\pi}\right)^n$ is a $\boxed{\text{Divergent Geometric series}}$ with $|r| = \left|\frac{6}{\pi}\right| = \frac{6}{\pi} > 1$.

Note: once we gain the Ratio Test in class, you can use the Ratio test here too.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{6}$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$ because $\displaystyle\lim_{n\to\infty} \frac{\pi}{6} = \frac{\pi}{6} \neq 0$

Note: The given series is also Divergent by the Geometric Series Test since $|r| = 1 \geq 1$.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{6^n} = \pi \sum_{n=1}^{\infty} \frac{1}{6^n}$ is a $\boxed{\text{Convergent Geometric series}}$ with $|r| = \left|\frac{1}{6}\right| = \frac{1}{6} < 1$.

Note: once we gain the Ratio Test in class, you can use the Ratio test here too.

16. $\displaystyle\sum_{n=1}^{\infty} \arctan(6n)$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$ because

$\displaystyle\lim_{n\to\infty} \arctan(6n \to \infty) = \frac{\pi}{2} \neq 0$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^6+1} \approx \sum_{n=1}^{\infty} \frac{1}{n^6+1} \approx \sum_{n=1}^{\infty} \frac{1}{n^6}$ which is a Convergent p-series with $p = 6 > 1$.

Bound Terms: $\dfrac{\sin^2 n}{n^6+1} < \dfrac{1}{n^6+1} < \dfrac{1}{n^6}$

Therefore, the Original Series Converges by the Comparison Test

18. $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{2}{n^6}\right)^{n^6}$

Diverges by n^{th} term Divergence Test because

$$\lim_{n\to\infty}\left(1-\frac{2}{n^6}\right)^{n^6} \overset{1^\infty}{} = \lim_{x\to\infty}\left(1-\frac{2}{x^6}\right)^{x^6} = e^{\lim\limits_{x\to\infty}\ln\left(\left(1-\frac{2}{x^6}\right)^{x^6}\right)}$$

$$= e^{\lim\limits_{x\to\infty} x^6\ln\left(1-\frac{2}{x^6}\right)} \overset{\infty\cdot 0}{=} e^{\lim\limits_{x\to\infty}\frac{\ln\left(1-\frac{2}{x^6}\right)}{\frac{1}{x^6}}}\ \frac{0}{0}$$

$$\overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\frac{\frac{1}{1-\frac{2}{x^6}}\left(\frac{6}{x^7}\right)}{-\frac{6}{x^7}}} = e^{\lim\limits_{x\to\infty}\frac{1}{1-\frac{2}{x^6}}(-2)}$$

$$= e^{(1)(-2)} = e^{-2} \neq 0$$