Find the MacLaurin Series representation for each of the following functions. State the Radius of Convergence for each series. You answer should be in sigma notation $\sum_{n=0}^{\infty}$.

Note: Here we will use substiution into out 6 known MacLaurin Series, as well as the known Radius of Convergence for each series.

1. $\dfrac{x^2}{1+5x} = x^2\left(\dfrac{1}{1+5x}\right) = x^2\left(\dfrac{1}{1-(-5x)}\right) = x^2\sum_{n=0}^{\infty}(-5x)^n$

$$= x^2\sum_{n=0}^{\infty}(-1)^n 5^n x^n = \boxed{\sum_{n=0}^{\infty}(-1)^n 5^n x^{n+2}}$$

Need $|-5x| = |5x| < 1 \Rightarrow |x| < \frac{1}{5}$. Here $\boxed{R = \frac{1}{5}}$

Recall, (finite) constant multiples will not change Convergence.

2. $x^7\sin(x^2) = x^7\sum_{n=0}^{\infty}\dfrac{(-1)^n\left(x^2\right)^{2n+1}}{(2n+1)!} = x^7\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{4n+2}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{4n+9}}{(2n+1)!}}$

Here $\boxed{R=\infty}$ for $\sin x$.

3. $x\arctan(3x) = x\sum_{n=0}^{\infty}\dfrac{(-1)^n(3x)^{2n+1}}{2n+1} = x\sum_{n=0}^{\infty}\dfrac{(-1)^n 3^{2n+1}x^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty}\dfrac{(-1)^n 3^{2n+1}x^{2n+2}}{2n+1}}$

Need $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$. Here $\boxed{R = \frac{1}{3}}$

4. $x^4 e^{-x^3} = x^4\sum_{n=0}^{\infty}\dfrac{\left(-x^3\right)^n}{n!} = x^4\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{3n}}{n!} = \boxed{\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{3n+4}}{n!}}$

Here $\boxed{R = \infty}$ for e^x.

5. $x^3 \ln(1+x^3) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^3\right)^{n+1}}{n+1} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+6}}{n+1}}$

Need $|x^3| < 1 \Rightarrow |x| < 1$. Here $\boxed{R = 1}$

6. $4x^2 \cos(4x) = 4x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!} = 4x^2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{2n+2}}{(2n)!}}$

Here $\boxed{R = \infty}$ for $\cos x$.

7. $\frac{x^3}{4+x} \quad x^3 \left(\frac{1}{4+x}\right) = \frac{x^3}{4} \left(\frac{1}{1+\frac{x}{4}}\right) = \frac{x^3}{4} \left(\frac{1}{1-\left(-\frac{x}{4}\right)}\right) = \frac{x^3}{4} \sum_{n=0}^{\infty} \left(-\frac{x}{4}\right)^n$

$$= \frac{x^3}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^n} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{4^{n+1}}}$$

Need $\left|-\frac{x}{4}\right| = \left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$. Here $\boxed{R = 4}$

8. $\frac{d}{dx}\left(\frac{x^3}{4+x}\right)$

We will reuse the derived series from 7 above.

$$\frac{d}{dx}\left(\frac{x^3}{4+x}\right) = \ldots = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{4^{n+1}} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (n+3) x^{n+2}}{4^{n+1}}}$$

The Radius remains unchanged after Differentiation. So $\boxed{R = 4}$ still.

9. $\displaystyle\int 4x^2 \arctan\left(4x^2\right)\, dx = \int 4x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (4x^2)^{2n+1}}{2n+1}\, dx$

$$= \int 4x^2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{4n+2}}{2n+1}\, dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+2} x^{4n+4}}{2n+1}\, dx$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+2} x^{4n+5}}{(2n+1)(4n+5)} + C}$$

Need $|4x^2| = |x^2| < \frac{1}{4} \Rightarrow |x| < \frac{1}{2}$. $R = \frac{1}{2}$ before Integration.

The Radius remains unchanged after Integration. Here $\boxed{R = \frac{1}{2}}$ still

10. $\displaystyle\frac{d}{dx}\left(x^2 \ln(1+5x)\right) = \frac{d}{dx}\left(x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{n+1}}{n+1}\right)$

$$= \frac{d}{dx}\left(x^2 \sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+1} x^{n+1}}{n+1}\right)$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+1} x^{n+3}}{n+1}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+1} (n+3) x^{n+2}}{n+1}}$$

Need $|5x| < 1 \Rightarrow |x| < \frac{1}{5}$. $R = \frac{1}{5}$ before Differentiation.

The Radius remains unchanged after Differentiation. Here $\boxed{R = \frac{1}{5}}$ still.

11. $$\int 5x^3e^{-5x^4}\ dx = \int 5x^3 \sum_{n=0}^{\infty} \frac{(-5x^4)^n}{n!}\ dx = \int 5x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^{4n}}{n!}\ dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+1} x^{4n+3}}{n!}\ dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+1} x^{4n+4}}{n!(4n+4)} + C}$$

Here $R = \infty$ for e^x. The Radius remains unchanged after Integration. So $\boxed{R = \infty}$ still.

12. $$\frac{d}{dx}\left(7x^2e^{7x}\right) = \frac{d}{dx}\left(7x^2 \sum_{n=0}^{\infty} \frac{(7x)^n}{n!}\right) = \frac{d}{dx}\left(7x^2 \sum_{n=0}^{\infty} \frac{(7x)^n}{n!}\right)$$

$$= \frac{d}{dx}\left(7x^2 \sum_{n=0}^{\infty} \frac{7^n x^n}{n!}\right) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{7^{n+1} x^{n+2}}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{7^{n+1}(n+2)x^{n+1}}{n!}}$$

Here $R = \infty$ for e^x. The Radius remains unchanged after Integration. So $\boxed{R = \infty}$ still.

13. $$\int x^3\ \cos(8x^4)\ dx = \int x^3\ \sum_{n=0}^{\infty} \frac{(-1)^n (8x^4)^{2n}}{(2n)!}\ dx = \int x^3\ \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n} x^{8n}}{(2n)!}\ dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n} x^{8n+3}}{(2n)!}\ dx = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n} x^{8n+4}}{(2n)!(8n+4)} + C}$$

Here $R = \infty$ for $\cos x$. The Radius remains unchanged after Integration. So $\boxed{R = \infty}$ still.

14. $$\frac{d}{dx}\left(6x^3\sin(6x^2)\right) = \frac{d}{dx}\left(6x^3\sum_{n=0}^{\infty}\frac{(-1)^n\left(6x^2\right)^{2n+1}}{(2n+1)!}\right)$$

$$= \frac{d}{dx}\left(6x^3\sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n+1}x^{4n+2}}{(2n+1)!}\right)$$

$$= \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n+2}x^{4n+5}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n+2}(4n+5)x^{4n+4}}{(2n+1)!}}$$

Here $R=\infty$ for $\sin x$. The Radius remains unchanged after Integration. So $\boxed{R=\infty}$ still.

15. Prove the MacLaurin Series formula for $\arctan x$.

We will derive it using substitution and integration.

$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1-(-x^2)}\,dx$$

$$= \int \sum_{n=0}^{\infty}\left(-x^2\right)^n\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} + \cancel{C}^{\,0} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}}$$

To solve for $+C$, first expand this equation

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + C$$

Test $x=0$ into both sides of the equation above.

Note that $x=0$ is in the Interval of Convergence for this series because it is the *Center* point of this power series.

$$\cancel{\arctan 0}^{\,0} = 0 - \frac{0^3}{3} + \frac{0^5}{5} - \frac{0^7}{7} + \frac{0^9}{9} - \ldots + C$$

That is, $0 = 0 - 0 + 0 - 0 + 0 - \ldots + C \Rightarrow C = 0$, Substitute above.

Finally, $\arctan x = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}$

16. Prove the MacLaurin Series formula for $\ln(1+x)$.

First option is to derive it using substitution and integration.

$$\begin{aligned}
\ln(1+x) &= \int \frac{1}{1+x}\, dx = \int \frac{1}{1-(-x)}\, dx \\
&= \int \sum_{n=0}^{\infty} (-x)^n \, dx = \int \sum_{n=0}^{\infty} (-1)^n x^n \, dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + \cancelto{0}{C} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}}
\end{aligned}$$

To solve for $+C$, first expand this equation

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots + C$

Test $x = 0$ into both sides of the equation above.

Note that $x = 0$ is in the Interval of Convergence for this series because it is the *Center* point of this power series.

$$\cancelto{0}{\ln 1} = 0 - \frac{0^2}{2} + \frac{0^3}{3} - \frac{0^4}{4} + \frac{0^5}{5} - \ldots + C$$

That is, $0 = 0 - 0 + 0 - 0 + 0 - \ldots + C \Rightarrow C = 0$, Substitute above.

Finally, $\ln(1+x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}}$

The second option is to use the Definition (*Chart Method*)

$$
\begin{array}{|ll|}
\hline
f(x) = \ln(1+x) & f(0) = \ln 1 = 0 \\
f'(x) = \dfrac{1}{1+x} = (1+x)^{-1} & f'(0) = 1 \\
f''(x) = -(1+x)^{-2} = -\dfrac{1}{(1+x)^2} & f''(0) = -1 \\
f'''(x) = 2(1+x)^{-3} = \dfrac{2}{(1+x)^3} & f'''(0) = 2 \\
f^{(4)}(x) = -6(1+x)^{-4} = -6\dfrac{6}{(1+x)^4} & f^{(4)}(0) = -6 \\
\vdots & \vdots \\
\hline
\end{array}
$$

MacLaurin Series Formula:

$$\cancelto{0}{f(0)} + \cancelto{1}{f'(0)}x + \frac{\cancelto{-1}{f''(0)}}{2!}x^2 + \frac{\cancelto{2}{f'''(0)}}{3!}x^3 + \frac{\cancelto{-6}{f^{(4)}(0)}}{4!}x^4 + \dots$$

$$= 0 + 1 \cdot x + \frac{(-1)}{2!}x^2 + \frac{2}{3!}x^3 + \frac{(-6)}{4!}x^4 + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

17. Prove the MacLaurin Series formula for $\sin x$.

The First option is to use the Definition (*Chart Method*)

$f(x) = \sin x$	$f(0) = \sin 0 = 0$
$f'(x) = \cos x$	$f'(0) = \cos 0 = 1$
$f''(x) = -\sin x$	$f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x$	$f'''(0) = -\cos 0 = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = \sin 0 = 0$
$\vdots$	$\vdots$

MacLaurin Series Formula:

$$\cancelto{0}{f(0)} + \cancelto{1}{f'(0)}x + \frac{\cancelto{0}{f''(0)}}{2!}x^2 + \frac{\cancelto{-1}{f'''(0)}}{3!}x^3 + \frac{\cancelto{0}{f^{(4)}(0)}}{4!}x^4 + \frac{\cancelto{1}{f^{(5)}(0)}}{5!}x^5 + \frac{\cancelto{0}{f^{(6)}(0)}}{6!}x^6 + \dots$$

$$= 0 + 1 \cdot x + 0 \cdot x^2 + \frac{(-1)}{3!}x^3 + 0 \cdot x^4 + \frac{1}{5!}x^5 + 0 \cdot x^6 + \frac{(-1)}{7!}x^7 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

The Second option is to use Differentiation.

$$\begin{aligned}
\sin x &= \frac{d}{dx}(-\cos x) = \frac{d}{dx}\left(-\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right) \\
&= \frac{d}{dx}\left[-\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)\right] = \frac{d}{dx}\left(-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots\right) \\
&= 0 + \frac{\cancel{2}x}{\cancel{2!}} - \frac{\cancel{4}x^3}{\cancel{4} \cdot 3!} + \frac{\cancel{6}x^5}{\cancel{6} \cdot 5!} - \dots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}
\end{aligned}$$

The Third option is to use Integration.

$$\sin x = \int \cos x \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!(2n+1)} + C = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cancel{C}^{\,0}$$

To solve for $+C$, first expand this equation

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + C$$

Test $x = 0$ into our equation above.

Note that $x = 0$ is in the Interval of Convergence for this series because it is the *Center* point of this power series.

$$\cancel{\sin 0}^{\,0} = 0 - 0 + 0 - 0 + \ldots + C \Rightarrow C = 0$$

Finally, $\sin x = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$