Exam 3 Fall 24 Answer Key

1(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (3x+5)^n}{(3n+5)^2 \cdot 4^n}$ Ratio Test

Converges by Ratio Test when

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}(3x+5)^{n+1}}{(3(n+1)+5)^2 \cdot 4^{n+1}}}{\dfrac{\cancel{(-1)^n}(3x+5)^n}{(3n+5)^2\ 4^n}} \right| = \lim_{n\to\infty} \underbrace{\left|\frac{(3x+5)^{n+1}}{(3x+5)^n}\right|}_{|3x+5|} \cdot \cancel{\left(\frac{3n+5}{3n+8}\right)^2}^{\,1} \cdot \frac{\cancel{4^n}}{\underbrace{4^{n+1}}_{\cancel{4^n}\cdot 4}} = \frac{|3x+5|}{4} < 1$$

$$\frac{|3x+5|}{4} < 1 \Rightarrow |3x+5| < 4 \Rightarrow -4 < 3x+5 < 4 \Rightarrow -9 < 3x < -1 \Rightarrow -3 < x < -\frac{1}{3}$$

$\quad -5 \quad -5 \quad -5$

Manually Check Convergence at Endpoints

Take $x=-3$ Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n [3(-3)+5]^n}{(3n+5)^2 \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-4)^n}{(3n+5)^2 \cdot 4^n} = \sum_{n=1}^{\infty} \frac{\underbrace{\cancel{(-1)^n(-1)^n}}_{(-1)^{2n} \to 1 \text{ Even}} \cancel{4^n}}{(3n+5)^2 \cdot \cancel{4^n}} = \sum_{n=1}^{\infty} \frac{1}{(3n+5)^2} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Convergent p-Series $p=2>1$

Bound Terms

$$\frac{1}{(3n+5)^2} \le \frac{1}{n^2}$$

$\Rightarrow$ Series Converges by CT

OR LCT Limit

$$\lim_{n\to\infty} \frac{\frac{1}{(3n+5)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(3n+5)^2} = \lim_{n\to\infty} \left(\frac{\cancel{n}^{\,1/3}}{\cancel{3n+5}}\right)^{2 \to 1/9} = \frac{1}{9} \text{ Finite Non-Zero}$$

$\Rightarrow$ Series Converges by LCT

Take $x = \frac{-1}{3}$ Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\cancel{3}(-\frac{1}{\cancel{3}})+5]^n}{(3n+5)^2\, 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{4^n}}{(3n+5)^2 \cancel{4^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+5)^2} \xrightarrow{AS} \sum_{n=1}^{n} \frac{1}{(3n+5)^2}$$

Already Shown above to be Convergent using CT or LCT

ACT

Original Series Converges by ACT

OR

1. Isolate $b_n = \dfrac{1}{(3n+5)^2} > 0$
2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\cancel{(3n+5)^2}_{\infty}} \to 0 = 0$
3. Terms Decreasing

$$b_{n+1} = \frac{1}{(3(n+1)+5)^2} = \frac{1}{(3n+8)^2} \le \frac{1}{(3n+5)^2} = b_n$$

Series Converges by AST

Finally, Interval of Convergence $I = \left[-3, -\frac{1}{3}\right]$

Radius of Convergence $R = \frac{4}{3}$

$[\quad -3 \quad\quad -\frac{1}{3} \quad]$

$-\frac{9}{3}$ Length $\frac{8}{3}$ $\hookrightarrow$ Half $= \frac{4}{3}$

1(b) Multiple choices

$$\sum_{n=1}^{\infty} n^n (x-8)^n$$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\overbrace{(n+1)^{n+1}}^{(n+1)^n(n+1)}(x-8)^{n+1}}{n^n\,(x-8)^n}\right| = \lim_{n\to\infty} \frac{\cancelto{e}{(n+1)^n}}{n^n}\,\cancelto{\infty}{(n+1)}\,|x-8| = \infty > 1$$

Diverges by Ratio Test unless $x = 8$ (when $L = 0 < 1$)

$\Rightarrow \boxed{I = \{8\}}$

$\Rightarrow R = 0$ not needed

OR // others will work ..

$$\sum_{n=1}^{\infty} (2n)!\,(x-8)^n$$

$$\sum_{n=1}^{\infty} (n!)^2 (x-8)^n$$

$$\sum_{n=1}^{\infty} n!\,(x-8)^n$$

$$\sum_{n=1}^{\infty} (3n)!\,n^n (x-8)^n$$

$\vdots$

justify

2(a) $$\ln(1+9x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (9x^2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 9^{n+1} x^{2n+2}}{n+1}$$

Need $|9x^2| < 1 \Rightarrow |x|^2 < \frac{1}{9} \Rightarrow |x| < \frac{1}{3} \Rightarrow R = \frac{1}{3}$

Recall:

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

2(b) $x^3 e^{-4x} = x^3 \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{n+3}}{n!}$

$R=\infty$

$R=\infty$

$R=\infty$

Recall:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$R=\infty$

2(c) $\frac{d}{dx}\left(8x^4 \sin(8x)\right) = \frac{d}{dx}\left(8x^4 \sum_{n=0}^{\infty} \frac{(-1)^n (8x)^{2n+1}}{(2n+1)!}\right) = \frac{d}{dx}\, 8x^4 \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+1} x^{2n+1}}{(2n+1)!}$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+2} x^{2n+5}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+2}(2n+5) x^{2n+4}}{(2n+1)!}$$

$R=\infty$ STILL After Differentiation

Recall:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

2(d) $\int \frac{x^2}{8+x^3}\,dx = \int x^2\left(\frac{1}{8+x^3}\right)dx = \int \frac{x^2}{8}\left(\frac{1}{1+\frac{x^3}{8}}\right)dx = \int \frac{x^2}{8}\left(\frac{1}{1-\left(-\frac{x^3}{8}\right)}\right)dx$

$$= \int \frac{x^2}{8} \sum_{n=0}^{\infty} \left(\frac{-x^3}{8}\right)^n dx = \int \frac{x^2}{8} \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{8^n}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{8^{n+1}}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{8^{n+1}(3n+3)} + C$$

Need $\left|\frac{-x^3}{8}\right| < 1 \Rightarrow |x|^3 < 8$

$\Rightarrow |x| < 2$

Recall: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$R=2$ STILL After Integration

3. $\dfrac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$16 \cdot 24 = 384$

$$e^{-1/2} = 1 + \left(-\tfrac{1}{2}\right) + \frac{\left(-\frac{1}{2}\right)^2}{2!} + \frac{\overset{1/8}{\left(-\frac{1}{2}\right)^3}}{3!} + \frac{\overset{1/16}{\left(-\frac{1}{2}\right)^4}}{4!} + \ldots$$

$$= 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} - \ldots$$

Make Sure to Connect Solution Throughout

$$\approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} = \frac{48 - 24 + 6 - 1}{48} = \frac{29}{48}$$ ← Estimate

Using the Alternating Series Estimation Theorem (A.S.E.T) we can Estimate the Full Sum using only the first four terms with error at most $\frac{1}{384} < \frac{1}{100}$ as desired.

4 a. $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n} \cdot \frac{\pi}{3}}{(2n+1)! \cdot \frac{\pi}{3}} = \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}$$

Flip

Recall:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \frac{3}{\pi} \overset{\frac{\sqrt{3}}{2}}{\sin\left(\frac{\pi}{3}\right)} = \frac{3\sqrt{3}}{2\pi}$$

4b. $$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \ldots = -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\right)$$

$$= -\ln(1+1) = -\ln 2$$

Recall:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

warning: $\ln(1 + (-1)) = \ln 0$ undefined won't work

4 c. $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^n (2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{2n}(2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{2}\right)^{2n}}{(2n)!} = \pi \overset{0}{\cos\left(\frac{\pi}{2}\right)} = \pi \cdot 0 = 0$$

Recall:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Don't Drop

4d $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(\ln 9)^n}{2^n \cdot n!} = - \sum_{n=0}^{\infty} \frac{(-1)^n(\ln 9)^n}{2^n\, n!} = -\sum_{n=0}^{\infty} \frac{\left(\frac{-\ln 9}{2}\right)^n}{n!} = -e^{-\frac{\ln 9}{2}}$

Recall:
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= -e^{-\frac{1}{2}\ln 9} = -\cancel{e^{\ln}}\left(9^{-\frac{1}{2}}\right) = -\frac{1}{9^{1/2}} = -\frac{1}{\sqrt{9}} = -\frac{1}{3}$$

Extra 1

4e. $4+4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\dots = 4\left(1+\underbrace{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots}_{\arctan 1}\right)$

Recall: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \dots$

$$= 4\left[1 + \cancel{\arctan(1)}^{\frac{\pi}{4}}\right] = 4\left(1+\frac{\pi}{4}\right) \text{ or } = 4+\pi$$

4f. $-\frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots = \left(\cancel{\cos\pi}^{-1}\right) - 1 = -1-1 = -2$

Recall:
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \longrightarrow \cos\pi = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \dots$

missing

5. $\displaystyle\lim_{x\to 0} \frac{1-\cos(2x)}{e^{-x}-1+x} = \lim_{x\to 0} \frac{1-\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\dots\right)}{1+(-x)+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\frac{(-x)^4}{4!}+\dots-1+x}$

$$= \lim_{x\to 0} \frac{\cancel{1}-\cancel{1}+\frac{2^2x^2}{2!}-\frac{2^4x^4}{4!}+\frac{2^6x^6}{6!}-\dots}{\cancel{1}-\cancel{x}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\dots-\cancel{1}+\cancel{x}}$$

Need all ...

$$= \lim_{x\to 0} \frac{\frac{2^2x^2}{2!}-\frac{2^4x^4}{4!}+\frac{2^6x^6}{6!}-\dots}{\frac{x^2}{2}-\frac{x^3}{3!}+\frac{x^4}{4!}-\dots} \quad \frac{1}{x^2} \quad \frac{1}{x^2}$$

Show All Algebra Steps

$$= \lim_{x\to 0} \frac{\frac{4}{2}-\cancel{\frac{2^4x^2}{4!}+\frac{2^6x^4}{6!}}^{\to 0}-\dots}{\frac{1}{2}-\cancel{\frac{x}{3!}+\frac{x^2}{4!}}^{\to 0}-\dots} = \frac{\cancel{\frac{4}{2}}^{2}}{\frac{1}{2}} = 4$$

5. continued Check answer with Optional L'Hôpital's Rule

$$\lim_{x\to 0} \frac{1-\cos(2x)}{e^{-x}-1+x} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{2\sin(2x)}{-e^{-x}+1} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{4\cos(2x)}{\cancel{-}+\, e^{-x}} = \frac{4}{1} = 4$$ Match!

6. $$\arctan(x^4) = \int \frac{4x^3}{1+x^8}\,dx = \int 4x^3\left(\frac{1}{1+x^8}\right)dx$$ Don't Drop dx's

$$= \int 4x^3\left(\frac{1}{1-(-x^8)}\right)dx = \int 4x^3 \sum_{n=0}^{\infty}(-x^8)^n\,dx$$

$$= \int 4x^3 \sum_{n=0}^{\infty}(-1)^n x^{8n}\,dx = \int 4\sum_{n=0}^{\infty}(-1)^n x^{8n+3}\,dx$$

$$= \cancel{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{\underset{\cancel{4}(2n+1)}{8n+4}} + C = \sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{2n+1} + \cancel{C}$$

Expand in Long Form to Solve for $+C$

$$\arctan(x^4) = \frac{x^4}{1} - \frac{x^{12}}{3} + \frac{x^{20}}{5} - \dots + C$$

Test the Center $x=0$ into both sides to solve for $+C$

$$\cancel{\arctan 0} = \cancel{0-0+0-\dots} + C \Rightarrow C = 0$$

Finally, $$\boxed{\arctan(x^4) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{2n+1}}$$ OR $$= 4\sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{8n+4}$$

Optional Check using "Substitution" into $\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$

$$\arctan(x^4) = \sum_{n=0}^{\infty}\frac{(-1)^n (x^4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{2n+1}$$ Match!