Exam #3 Final Answers Fall 2021

1(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (6x-5)^n}{n^6 \cdot 7^n}$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}}(6x-5)^{n+1}}{(n+1)^6 \cdot 7^{n+1}}}{\frac{\cancel{(-1)^n}(6x-5)^n}{n^6 \cdot 7^n}} \right| = \lim_{n\to\infty} \left|\frac{\cancel{(6x-5)^{n+1}}}{\cancel{(6x-5)^n}}\right| \cdot \frac{n^6}{(n+1)^6} \cdot \frac{\cancel{7^n}}{7^{n+1}}$$

$|6x-5|$ $\qquad \left(\frac{n}{n+1}\right)^6 \to 1$ $\qquad \cancel{7^n}\,7$

$$= \frac{|6x-5|}{7} < 1 \quad \text{Converges by Ratio Test when}$$

$$\Rightarrow |6x-5| < 7 \Rightarrow -7 < 6x-5 < 7$$

$+5 \quad +5 \quad +5$

$$-2 < 6x < 12$$

$$-\frac{1}{3} < x < 2$$

Manually Check Convergence at Endpoints

Take $x = -\frac{1}{3}$. Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n \left[6\left(-\frac{1}{3}\right) - 5\right]^n}{n^6 \cdot 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-7)^n}{n^6 \cdot \cancel{7^n}} = \sum_{n=1}^{\infty} \frac{\cancel{(-1)^{2n}}}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^6}$$

-2 $\qquad (-1)^n\, \cancel{7^n}$ $\qquad$ even $\to 1$

Converges, p-Series $p = 6 > 1$

OR

$$\sum_{n=1}^{\infty} \frac{[(-1)(-7)]^n}{n^6\, 7^n} = \sum_{n=1}^{\infty} \frac{\cancel{7^n}}{n^6 \cdot \cancel{7^n}} = \sum_{n=1}^{\infty} \frac{1}{n^6}$$

Take $x = 2$. Series becomes

Two Main Choices

either

$$\sum_{n=1}^{\infty} \frac{(-1)^n [6(2) - 5]^n}{n^6 \cdot 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{7^n}}{n^6 \cdot \cancel{7^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^6} \quad \text{A.S.} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} \text{ Converges p-Series } p = 6 > 1$$

OR

Converges by AST

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^6} \text{ Converges by A.C.T.}$$

① $b_n = \frac{1}{n^6} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^6} = 0$ $\qquad (n^6 \to \infty)$

③ Terms Decreasing

$$b_{n+1} = \frac{1}{(n+1)^6} \le \frac{1}{n^6} = b_n$$

OR $f(x) = \frac{1}{x^6}$ has $f'(x) = \frac{-6}{x^7} < 0$ for $x > 0$

Finally, $I = \left[-\frac{1}{3}, 2\right]$

$R = \frac{7}{6}$

Total $= \frac{7}{3}$ $\quad \hookrightarrow$ Half $= \frac{7}{6}$

$\frac{1}{3}$ $\qquad 2 = \frac{6}{3}$

$-\frac{1}{3} \quad 0 \quad 2$

1(b) $\sin x \hookrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{\cancel{(-1)^n} x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{\overbrace{x^{\cancel{2n+3}}}^{|x^2|}}{\cancel{x^{2n+1}}} \right| \cdot \frac{\cancel{(2n+1)!}}{\underbrace{(2n+3)!}_{(2n+3)(2n+2)\cancel{(2n+1)!}}}$$

$$= \lim_{n\to\infty} \frac{|x|^2 \to 0}{\underbrace{(2n+3)(2n+2)}_{\to\infty}} = 0 < 1 \quad \text{always}$$

Converges by Ratio Test for all x in $\mathbb{R}$.

$\Rightarrow I = (-\infty, \infty)$ not needed here

$\Rightarrow R = \infty$

1(c) Multiple choices

$$\sum_{n=1}^{\infty} n^n (x-8)^n$$

Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\overbrace{(n+1)^{n+1}}^{(n+1)^n(n+1)} (x-8)^{n+1}}{n^n (x-8)^n} \right| = \lim_{n\to\infty} \underbrace{\frac{(n+1)^n}{n^n}}_{\to e} \underbrace{(n+1)}_{\to\infty} |x-8| = \infty > 1$$

Diverges by Ratio Test unless $x = 8$ (when $L = 0 < 1$)

$\Rightarrow I = \{8\}$

$\Rightarrow R = 0$ not needed here

OR, others will work ...

$$\sum_{n=1}^{\infty} (2n)!\,(x-8)^n$$

$$\sum_{n=1}^{\infty} (n!)^2 (x-8)^n$$

$$\sum_{n=1}^{\infty} n!\,(x-8)^n$$

$$\sum_{n=1}^{\infty} (3n)!\,n^n (x-8)^n$$

$\vdots$

justify here

2(a) $\frac{d}{dx}\left(7x^4\arctan(7x)\right) = \frac{d}{dx}\left(7x^4\sum_{n=0}^{\infty}\frac{(-1)^n(7x)^{2n+1}}{2n+1}\right)$

Substitution

Need $|7x|<1$

$\hookrightarrow |x|<1/7$

$$= \frac{d}{dx}\left(7x^4\sum_{n=0}^{\infty}\frac{(-1)^n 7^{2n+1}x^{2n+1}}{2n+1}\right)$$

$R=\frac{1}{7}$

$$= \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{(-1)^n 7^{2n+2}x^{2n+5}}{2n+1}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 7^{2n+2}(2n+5)x^{2n+4}}{2n+1}$$

$R=\frac{1}{7}$ STILL after Derivative

2(b) $\int\frac{x^3}{5+x}\,dx = \int x^3\left(\frac{1}{5+x}\right)dx = \int\frac{x^3}{5}\left(\frac{1}{1+x/5}\right)dx = \int\frac{x^3}{5}\left[\frac{1}{1-\left(-\frac{x}{5}\right)}\right]dx$

Geometric

Need $\left|\frac{x}{5}\right|<1 \Rightarrow |x|<5 \Rightarrow R=5$

$$= \int\frac{x^3}{5}\sum_{n=0}^{\infty}\left(\frac{-x}{5}\right)^n dx = \int\frac{x^3}{5}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{5^n}\,dx = \int\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{5^{n+1}}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+4}}{5^{n+1}\cdot(n+4)} + C$$

$R=5$ STILL after integration

3. $\lim_{x\to0}\frac{e^x-e^{-x}-2x}{x^3} = \lim_{x\to0}\frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots\right)-2x}{x^3}$

$$= \lim_{x\to0}\frac{\cancel{1}+x+\cancel{\frac{x^2}{2!}}+\frac{x^3}{3!}+\cancel{\frac{x^4}{4!}}+\cdots\cancel{-1}+x-\cancel{\frac{x^2}{2!}}+\frac{x^3}{3!}-\cancel{\frac{x^4}{4!}}+\cdots\cancel{-2x}}{x^3}$$

$$= \lim_{x\to0}\frac{\frac{2x^3}{3!}+\frac{2x^5}{5!}+\cdots}{x^3}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

OR, Split-Split ...

3 (Continued)

$$= \lim_{x \to 0} \frac{\frac{2}{6} + \frac{2x^2}{5!} + \cdots}{1} = \frac{1}{3}$$

$\frac{2x^2}{5!} \to 0$, all x-terms $\to 0$ ....

L'Hôpital's Rule

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x^3} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{3x^2} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{e^x - e^{-x}}{6x} \overset{0/0}{=} \lim_{x\to0} \frac{\overset{1}{\cancel{e^x}} + \overset{1}{\cancel{e^{-x}}}}{6} = \frac{2}{6} = \frac{1}{3}$$ Match!

Note: $e^x - e^{-x} = 2\sinh x$ by chance

4. Recall: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

$$\ln\left(\tfrac{3}{2}\right) = \ln\left(1 + \tfrac{1}{2}\right) = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^4}{4} + \cdots$$

$$= \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \cdots$$

Keep

$$\approx \frac{12}{24} - \frac{3}{24} + \frac{1}{24} = \frac{10}{24} = \frac{5}{12}$$ ← Estimate

Using A.S.E.T. this estimate yields Error at most $\frac{1}{64} < \frac{1}{50}$ as desired.

5(a) $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^{n-1}(2n+1)!} \cdot \frac{9}{9} = 9 \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{\underset{3^{2n}}{9^n}(2n+1)!} = 9 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n+1)!} \frac{\frac{\pi}{3}}{\frac{\pi}{3}}$$

or $9^n \cdot 9^{-1}$

$$= \frac{27}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \frac{27}{\pi} \overset{\frac{\sqrt{3}}{2}}{\cancel{\sin\left(\frac{\pi}{3}\right)}} = \frac{27\sqrt{3}}{2\pi}$$

Recall: $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

arctan 1

5(b) $$4 + 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = 4\left(1 + \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)\right)$$

$$= 4(1 + \arctan 1) = 4\left(1 + \frac{\pi}{4}\right) = 4 + \pi$$

Recall: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

$$5(c)\ \sum_{n=0}^{\infty} \frac{(-1)^n (\pi^2)^{n+1}}{(\sqrt{6})^{4n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+2}}{6^{2n}(2n)!} = \pi^2 \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$$

extra π^2

$$= \pi^2 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \pi^2 \cos\left(\pi/6\right) = \frac{\sqrt{3}\,\pi^2}{2}$$

$\frac{\sqrt{3}}{2}$

Recall: $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$5(d)\ \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (\ln 8)^n}{3^n\, n!} = -\sum_{n=0}^{\infty} \frac{(-1)^n (\ln 8)^n}{3^n \cdot n!} = -\sum_{n=0}^{\infty} \frac{\left(\frac{-\ln 8}{3}\right)^n}{n!} = -e^{-\frac{\ln 8}{3}} = -e^{-\frac{1}{3}\ln 8}$$

$$= -\cancel{e}^{\cancel{\ln}(8^{-1/3})} = -\frac{1}{8^{1/3}} = -\frac{1}{\sqrt[3]{8}} = -\frac{1}{2}$$

2

Recall: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

OR, $-e^{-\frac{1}{3}\ln 8} = -e^{-\frac{1}{3}\ln(2^3)} = -e^{-\frac{\cancel{3}}{\cancel{3}}\ln 2} = -\frac{1}{\cancel{e}^{\cancel{\ln} 2}} = -\frac{1}{2}$

extra

$$5(e)\ -1-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\dots = -1-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots\right)$$

$\ln(1+1)$

$$= -1-\ln 2$$

Recall. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ opposite signs here $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\dots$

then extra -1 out front

$$5(f)\ 1-\frac{1}{e}+\frac{1}{e^2\cdot 2!}-\frac{1}{e^3\cdot 3!}+\frac{1}{e^4\cdot 4!}-\dots = 1+\left(-\frac{1}{e}\right)+\frac{\left(-\frac{1}{e}\right)^2}{2!}+\frac{\left(-\frac{1}{e}\right)^3}{3!}+\dots$$

$$= e^{-1/e}$$

Recall: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Geometric

6. $\arctan x = \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1-(-x^2)}\,dx = \int \sum_{n=0}^{\infty} (-x^2)^n\,dx$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + \cancel{C} \to 0$$

Test $x=0$ to solve for $+C$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + C$$

$$\cancel{\arctan 0} \to 0 = \cancel{0 - 0 + 0 - \cdots} \to 0 + C \Rightarrow C = 0$$

Finally, $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

OPTIONAL BONUS

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3n+2)}{(n+1)(2n+1)\,3^{n+1}}$$ looks like $$\sum_{n=0}^{\infty} \frac{(-1)^n (3n+2)\,(x^2)^{n+1}}{(n+1)(2n+1)}$$ where $x = \frac{1}{\sqrt{3}}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (3n+2)\,x^{2n+2}}{(n+1)(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n[(n+1)+(2n+1)]\,x^{2n+2}}{(n+1)(2n+1)}$$ split-split

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{(n+1)}\,x^{2n+2}}{\cancel{(n+1)}(2n+1)} + \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{(2n+1)}}{(n+1)\cancel{(2n+1)}}\,x^{2n+2}$$

$$= x\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{n+1}}{n+1}$$ Reverse Pattern Match!

$$= x\arctan x + \ln(1+x^2)$$

Finally, substituting $x = \frac{1}{\sqrt{3}}$ yields

$$= \frac{1}{\sqrt{3}}\,\cancel{\arctan\frac{1}{\sqrt{3}}}^{\pi/6} + \ln\left|1+\left(\frac{1}{\sqrt{3}}\right)^2\right|^{4/3} = \frac{\pi}{6\sqrt{3}} + \ln\left(\frac{4}{3}\right)$$