Math 121 Exam 2 Spring 2024 Answer Key

1(a) $$\int_0^e x^2 \ln(x^2)\,dx = \lim_{t\to 0^+} \int_t^e x^2 \ln(x^2)\,dx = \lim_{t\to 0^+} \frac{x^3}{3}\cdot \ln(x^2)\Big|_t^e - \frac{2}{3}\int_t^e x^2\,dx$$

$$\boxed{\begin{aligned} u &= \ln(x^2) & dv &= x^2\,dx \\ du &= \frac{1}{x^2}(2x)\,dx & v &= \frac{x^3}{3} \\ &= \frac{2}{x}\,dx \end{aligned}}$$

$$= \lim_{t\to 0^+} \frac{x^3}{3}\cdot \ln(x^2)\Big|_t^e - \frac{2x^3}{9}\Big|_t^e$$

$$= \lim_{t\to 0^+} \frac{e^3}{3}\cdot \cancel{\ln(e^2)}^{2} - \cancel{\frac{t^3}{3}\cdot\ln(t^2)}^{\overset{0\cdot(-\infty)}{0}} - \left(\frac{2e^3}{9} - \cancel{\frac{2t^3}{9}}^{0}\right)$$

See (*)

$$= \frac{2e^3}{3} - \frac{2e^3}{9} = \frac{6e^3}{9} - \frac{2e^3}{9} = \frac{4e^3}{9}$$ Converges

Match

(*) $$\lim_{t\to 0^+} \overset{0\cdot(-\infty)}{t^3\cdot\ln(t^2)} = \lim_{t\to 0^+} \frac{\ln(t^2)}{\frac{1}{t^3}\ t^{-3}} \overset{\frac{-\infty}{\infty}}{\underset{\text{L'H}}{=}} \lim_{t\to 0^+} \frac{\cancel{\frac{1}{t^2}\cdot(2t)}\ t}{\frac{-3}{t^4}}\ \frac{-t^4}{3} = \lim_{t\to 0^+} \frac{-2t^{\cancel{4}\,3}}{3\cancel{t}} = \lim_{t\to 0^+} -\frac{2\cancel{t^3}^{0}}{3} = 0$$

cancel

Key Note: ln 0 is undefined, so must "sneak attack" 0 using Limit 0^+

1(b) $$\int_e^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_e^t (\ln x)\cdot x^{-2}\,dx \overset{\text{IBP}}{=} \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_e^t + \int_e^t x^{-2}\,dx$$

$$\boxed{\begin{aligned} u &= \ln x & dv &= x^{-2}\,dx \\ du &= \frac{1}{x}\,dx & v &= \frac{x^{-1}}{-1} = -\frac{1}{x} \end{aligned}}$$

$$= \lim_{t\to\infty} -\frac{\ln x}{x}\Big|_e^t - \frac{1}{x}\Big|_e^t$$

$$= \lim_{t\to\infty} \overset{\frac{\infty}{\infty}}{-\frac{\ln t}{t}} + \frac{\cancel{\ln e}^{1}}{e} - \cancel{\frac{1}{t}}^{0} + \frac{1}{e}$$

$$\underset{\text{L'H}}{=} \lim_{t\to\infty} \frac{-\cancel{\frac{1}{t}}^{0}}{1} + \frac{1}{e} + \frac{1}{e} = \frac{2}{e}$$ Converges

Match!

1(c) $\displaystyle\int_{-\infty}^{-3} \frac{8-x}{x^2+2x+5}\,dx = \lim_{t\to-\infty}\int_{t}^{-3} \frac{8-x}{x^2+2x+5}\,dx \overset{\text{Complete Square}}{=} \lim_{t\to-\infty}\int_{t}^{-3} \frac{8-x}{(x+1)^2+4}\,dx$

$(x+1)^2 = x^2+2x+1$
$\quad +4 \qquad\qquad +4$

$$= \lim_{t\to-\infty}\int_{t+1}^{-2} \frac{8-(u-1)}{u^2+4}\,du = \lim_{t\to-\infty}\int_{t+1}^{-2} \frac{9-u}{u^2+4}\,du$$

$u = x+1 \Rightarrow x = u-1$
$du = dx$

$x = t \Rightarrow u = t+1$
$x = -3 \Rightarrow u = -3+1 = -2$

$$= \lim_{t\to-\infty}\int_{t+1}^{-2} \frac{9}{u^2+4} - \frac{u}{u^2+4}\,du$$

$$= \lim_{t\to-\infty} \frac{9}{2}\arctan\left(\frac{u}{2}\right) - \frac{1}{2}\ln|u^2+4| \Bigg|_{t+1}^{-2}$$

$$= \lim_{t\to-\infty} \underbrace{\frac{9}{2}\arctan\left(\overset{-1}{\frac{-2}{2}}\right)}_{\text{Finite}}^{-\frac{\pi}{4}} - \underbrace{\frac{1}{2}\ln 8}_{\text{Finite}} - \left(\underbrace{\frac{9}{2}\arctan\left(\overset{-\infty}{\frac{t+1}{2}}\right)}_{\text{Finite}}^{-\frac{\pi}{2}} - \overset{\infty}{\frac{1}{2}\ln|\overset{\infty}{(\underset{-\infty}{t}+1)^2}+4|}\right)$$

$= \infty$ Diverges
Matches!

Finish all pieces
Find all arctan values.

1(d) $\displaystyle\int_{-4}^{-3} \frac{8-x}{x^2+2x-8}\,dx = \int_{-4}^{-3} \frac{8-x}{(x-2)(x+4)}\,dx = \lim_{t\to-4^+}\int_{t}^{-3} \frac{8-x}{(x-2)(x+4)}\,dx$

PFD

$$\cancel{(x-2)(x+4)}\left(\frac{8-x}{\cancel{(x-2)(x+4)}} = \frac{A}{x-2} + \frac{B}{x+4}\right)(x-2)(x+4)$$

$8-x = A(x+4)+B(x-2)$
$\quad = Ax+4A+Bx-2B$
$\quad = (A+B)x+(4A-2B)$

Conditions

- $A+B=-1 \Rightarrow A=-1-B$
- $4A-2B=8 \quad 4(-1-B)-2B=8$
  $-4-4B-2B=8$
  $-6B=12$
  $B=-2$
  $\Downarrow$
  $A=-1-(-2)=1$

$$= \lim_{t\to-4^+}\int_{t}^{-3} \frac{1}{x-2} - \frac{2}{x+4}\,dx$$

$$= \lim_{t\to-4^+} \ln|x-2| - 2\ln|x+4| \Bigg|_{t}^{-3}$$

Justify size!

$$= \lim_{t\to-4^+} \underbrace{\ln|-5|}_{\text{Finite}}^{\ln 5} - 2\ln\overset{0}{|-1|} - \left(\underbrace{\ln|\overset{-4}{t}-2|}_{\text{Finite}}^{\ln 6} - \underbrace{2\ln|\overset{-4^+}{t}+4|}_{\text{blows up}}^{-\infty,\ 0^+}\right)$$

$= -\infty$ Diverges
Match!

Simplify $\ln 1 = 0$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4n+7} \quad \hookrightarrow \quad \int_1^{\infty} \frac{1}{x^2+4x+7}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4x+7}\,dx$

Complete Square

$$= \lim_{t\to\infty} \int_1^t \frac{1}{(x+2)^2+3}\,dx$$

$\boxed{\begin{array}{l} u = x+2 \\ du = dx \end{array}}$

$\boxed{\begin{array}{l} x=1 \Rightarrow u=3 \\ x=t \Rightarrow u=t+2 \end{array}}$

$$= \lim_{t\to\infty} \int_3^{t+2} \frac{1}{u^2+3}\,du$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_3^{t+2}$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \left(\underbrace{\arctan\left(\frac{t+2}{\sqrt{3}}\right)}_{\frac{\pi}{2}} - \underbrace{\arctan\left(\frac{3}{\sqrt{3}}\right)}_{\frac{\pi}{3}}\right)$$

$\infty \qquad \sqrt{3}$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{3\pi}{6} - \frac{2\pi}{6}\right) = \frac{\pi}{6\sqrt{3}}$$

Integral Converges

$\Rightarrow$ $\boxed{\text{Series Converges by Integral Test}}$

3(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^5+8}{8n^5+1}$ $\boxed{\text{Diverges by nTDT}}$ because

$$\lim_{n\to\infty} \frac{n^5+8}{8n^5+1} \cdot \frac{\frac{1}{n^5}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{1+\cancel{\frac{8}{n^5}}^{\,0}}{8+\cancel{\frac{1}{n^5}}^{\,0}} = \frac{1}{8} \neq 0$$

3(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+5)^8}{\ln(n+5)}$ $\boxed{\text{Diverges by nTDT}}$ because

$$\lim_{n\to\infty} \frac{(n+5)^8}{\ln(n+5)} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{(x+5)^8}{\ln(x+5)} \stackrel{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{8(x+5)^7}{\frac{1}{x+5}} = \lim_{x\to\infty} 8\,\cancel{(x+5)^8}^{\,\infty} = \infty \neq 0$$

$x+5$

3(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^8} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{n^8}$ Converges p-Series $p=8>1$

$\Swarrow$

Original Series $\boxed{\text{Converges by the Absolute Convergence Test}}$ (ACT)

OR

1. Pick $b_n = \frac{1}{n^8} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \cancel{\frac{1}{n^8}}^{\,0} = 0$ ($n^8 \to \infty$)
3. Terms Decreasing

$b_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = b_n$

$\boxed{\text{Original Series Converges by the Alternating Series Test}}$

3(d) $\sum_{n=1}^{\infty} \frac{\ln 5}{(n+5)^8} + \frac{(-1)^n 8}{5^{2n+1}} \overset{\text{split}}{=} \sum_{n=1}^{\infty} \frac{\ln 5}{(n+5)^8} + \sum_{n=1}^{\infty} \frac{(-1)^n 8}{5^{2n+1}}$

$\ln 5 \sum_{n=1}^{\infty} \frac{1}{(n+5)^8}$ ☆ $+$ $-\frac{8}{5^3} + \frac{8}{5^5} - \frac{8}{5^7} + \dots$

$r = -\frac{1}{5^2}$

Converges by CT

Constant Multiple of Convergent Series is Convergent

Converges by Geometric Series Test with $|r| = \left|-\frac{1}{5^2}\right| = \frac{1}{25} < 1$

☆ $\sum_{n=1}^{\infty} \frac{1}{(n+5)^8} \approx \sum_{n=1}^{\infty} \frac{1}{n^8}$

Convergent p-Series $p = 8 > 1$

Bound Terms

$\frac{1}{(n+5)^8} \leq \frac{1}{n^8}$

$\hookrightarrow \sum_{n=1}^{\infty} \frac{1}{(n+5)^8}$ Converges by CT

OR LCT

Original Series Converges because the Sum of Two Convergent Series is Convergent

3(e) $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \dots = - \sum_{n=0}^{\infty} \frac{1}{n}$

Note: not Alternating here

Constant Multiple of the Divergent Harmonic p-Series with $p = 1$ is Divergent

4(a) Diverges by nTDT using L'H Rule

Examples: $\sum_{n=2}^{\infty} \frac{n^2}{\ln n}$ $\quad \sum_{n=2}^{\infty} \frac{e^n}{\ln n}$ $\quad \sum_{n=1}^{\infty} \frac{e^n}{n^2}$

$\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$ $\quad \sum_{n=1}^{\infty} \left(1 - \frac{5}{n^2}\right)^{n^2}$

Continue on to prove your case.

Switch to Related function in x for L'H

4(b) Converges by Comparison Test. Need "Smaller than Converge" Comparison

Examples: $\sum_{n=1}^{\infty} \frac{n^2}{n^9+5}$ $\quad \sum_{n=1}^{\infty} \frac{1}{9^n+5}$ $\quad \sum_{n=1}^{\infty} \frac{3^n}{7^n+6}$ $\quad \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^5+4}$ $\quad \sum_{n=1}^{\infty} \frac{\cos^2 n}{8^n+1}$

Sample. $\sum_{n=1}^{\infty} \frac{1}{n^6+9} \approx \sum_{n=1}^{\infty} \frac{1}{n^6}$ Converges p-Series $p = 6 > 1$

Bound Terms

$\frac{1}{n^6+9} \leq \frac{1}{n^6}$

$\Rightarrow$ Original Series also Converges by Comparison Test

4(c) Absolutely Convergent by Ratio Test

Examples. $\sum_{n=1}^{\infty} \frac{1}{n!}$ $\quad \sum_{n=1}^{\infty} \frac{e^n}{n!}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^n}$ $\quad \sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^n (2n)!}$ $\quad \sum_{n=1}^{\infty} \frac{n^6 n^n (n!)^2}{(3n)!}$

Simplest?!

Sample: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \right| = \lim_{n\to\infty} \frac{\overbrace{(n+1)!}^{\cancel{(n+1)}\cdot \cancel{n!}}}{\cancel{n!}} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n \cdot \cancel{(n+1)}}} = \lim_{n\to\infty} \cancel{\frac{n^n}{(n+1)^n}} \xrightarrow{} \frac{1}{e} = \frac{1}{e} < 1$$

$\Rightarrow$ $\boxed{\text{Series Converges Absolutely by Ratio Test}}$

4(d) Alternating Series Convergent by ACT

Examples: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6+1}$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^8+7}$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n n^6}{n^9+5}$

Sample:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5+n^4+n^3+n^2+n+1} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty} \frac{1}{n^5+n^4+n^3+n^2+n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n^5}$$

Converges p-Series
$p=5>1$

<u>Bound Terms</u>

$$\frac{1}{n^5+n^4+n^3+n^2+n+1} \leq \frac{1}{n^5}$$

Note: Limit Comparison also Works here

$\Rightarrow$ Absolute Series also Converges by the Comparison Test

ACT $\Leftarrow$

Original Series

Converges by the Absolute Convergence Test

5(a) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^5+5n+8}{n^8+5}\right) \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{n^5+5n+8}{n^8+5} \approx \sum_{n=1}^{\infty} \frac{n^5}{n^8} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges p-Series $p=3>1$

$$\lim_{n\to\infty} \frac{\frac{n^5+5n+8}{n^8+5}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^8+5n^4+8n^3}{n^8+5} \frac{\frac{1}{n^8}}{\frac{1}{n^8}} = \lim_{n\to\infty} \frac{1+\cancel{\frac{5}{n^4}}+\cancel{\frac{8}{n^5}}}{1+\cancel{\frac{5}{n^8}}} = 1$$ Finite Non-Zero

$\Rightarrow$ Absolute Series also Converges by the Limit Comparison Test

$\Rightarrow$ Original Series is Absolutely Convergent by Definition

note: no need to test original series

5(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^5 \cdot n^n \cdot n!}{(2n+1)!}$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}}(n+1)^5 (n+1)^{n+1} (n+1)!}{(2(n+1)+1)!}}{\frac{\cancel{(-1)^n} n^5 n^n n!}{(2n+1)!}} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \cdot \frac{(n+1)^{n+1}}{n^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{\cancel{(2n+1)!}}{(2n+3)!}$$

$(n+1)^{n+1} = (n+1)^n (n+1)$, $(n+1)! = (n+1)n!$, $(2n+3)! = \boxed{(2n+3)(2n+2)}\cancel{(2n+1)!}$

$\left(1+\frac{1}{n}\right)^5$ or $\left(\frac{n+1}{n} \frac{\frac{1}{n}}{\frac{1}{n}}\right)^5 \to \left(\frac{1+\frac{1}{n}}{1}\right)^5 \to 1$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \left(\frac{n+1}{2n+3}\frac{\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{\cancel{n+1}}{2n+2}\right)$$

$2n+2 = 2\cancel{(n+1)}$ Don't Drop

$$= \lim_{n\to\infty} \frac{e}{2}\left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right) = \frac{e}{4} < 1$$

$\Rightarrow$ the Series Converges Absolutely by the Ratio Test

$$5(c)\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{5n+8} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{5n+8} \approx \sum_{n=1}^{\infty} \frac{1}{n}$$ Diverges p-Series $p=1$

$$\lim_{n\to\infty} \frac{\frac{1}{5n+8}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+8}\cdot\frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{5+\frac{8}{n}} = \frac{1}{5}$$ Finite Non-Zero

$\Rightarrow$ Absolute Series also Diverges by the Limit Comparison Test

AST

1. Pick $b_n = \frac{1}{5n+8} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{5n+8} = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{\underbrace{5(n+1)+8}_{5n+13}} \leq \frac{1}{5n+8} = b_n$$

Original Series Converges by the Alternating Series Test

OR show Related Function

$f(x) = \frac{1}{5x+8}$ has $f'(x) = \frac{-5}{(5x+8)^2} < 0$

Original Series is Conditionally Convergent by Definition