Integrals

1.

$$\int_3^\infty \frac{1}{x^2-4x+7}\,dx = \lim_{t\to\infty}\int_3^t \frac{1}{(x-2)^2+3}\,dx = \lim_{t\to\infty}\int_1^{t-2}\frac{1}{u^2+3}\,du$$

Complete Square

$b^2-4ac = 16-4(1)(7) = \ominus$

$(x-2)^2 = x^2-4x+4$

$u = x-2$	$x=3 \Rightarrow u=1$
$du = dx$	$x=t \Rightarrow u=t-2$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Big|_1^{t-2}$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left[\arctan\left(\frac{t-2}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right]$$

(arrows: $\arctan \to \frac{\pi}{2}$ with $t-2 \to \infty$; $\arctan \to \frac{\pi}{6}$)

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{2}-\frac{\pi}{6}\right] = \frac{1}{\sqrt{3}}\left[\frac{2\pi}{6}\right] = \boxed{\frac{\pi}{3\sqrt{3}}} \qquad \text{Converges}$$

2.

$$\int_e^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_1^{\ln t}\frac{1}{u^3}\,du = \lim_{t\to\infty}\frac{-1}{2u^2}\Big|_1^{\ln t}$$

(note: $\frac{1}{u^3} = u^{-3}$)

$u = \ln x$	$x=e \Rightarrow u=\ln e = 1$
$du = \frac{1}{x}dx$	$x=t \Rightarrow u=\ln t$

$$= \lim_{t\to\infty}\; -\frac{1}{2(\ln t)^2} + \frac{1}{2} = \boxed{\frac{1}{2}} \qquad \text{Converges}$$

(annotations: $\to 0$ as $\ln t \to \infty$)

3.

$$\int_0^\infty \frac{1}{(x-2)(2x+5)}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{(x+2)(2x+5)}\,dx \overset{\text{PFD}}{=} \lim_{t\to\infty}\int_0^t \frac{1}{x+2} - \frac{2}{2x+5}\,dx$$

P.F.D.

$$(x+2)(2x+5)\left[\frac{1}{(x+2)(2x+5)} = \frac{A}{x+2} + \frac{B}{2x+5}\right](x+2)(2x+5)$$

$$1 = A(2x+5) + B(x+2)$$
$$= 2Ax + 5A + Bx + 2B$$
$$= (2A+B)x + (5A+2B)$$

$$= \lim_{t\to\infty} \ln|x+2| - \ln|2x+5|\Big|_0^t$$

$$= \lim_{t\to\infty} \ln|t+2| - \ln|2t+5| - \ln 2 + \ln 5$$

$$= \lim_{t\to\infty} \ln\left|\frac{t+2}{2t+5}\frac{\;^{1/t}}{\;_{1/t}}\right| - \ln 2 + \ln 5$$

Conditions
- $2A + B = 0 \Rightarrow B = -2A$
- $5A + 2B = 1 \quad 5A + 2(-2A) = 1$
$$A = 1 \Rightarrow B = -2$$

$$= \lim_{t \to \infty} \ln \left| \frac{1 + \frac{3}{t}}{2 + \frac{5}{t}} \right| - \ln 2 + \ln 5$$

$$= \ln\left(\tfrac{1}{2}\right) - \ln 2 + \ln 5 \qquad \text{Converges}$$

$$= \ln 1 - \ln 2 - \ln 2 + \ln 5 = \boxed{\ln\left(\tfrac{5}{4}\right)}$$
$$\underbrace{-2\ln 2}_{} \quad -\ln(2^2)$$
$$\searrow -\ln 4$$

4. $\displaystyle \int_7^\infty \frac{1}{x^2 - 8x + 19} dx = \lim_{t \to \infty} \int_7^t \frac{1}{x^2 - 8x + 19} dx = \lim_{t \to \infty} \int_7^t \frac{1}{(x-4)^2 + 3} dx$

Complete Square

$\boxed{u = x - 4 \\ du = dx}$ $\boxed{x = 7 \Rightarrow u = 3 \\ x = t \Rightarrow u = t - 4}$ $\displaystyle = \lim_{t \to \infty} \int_3^{t-4} \frac{1}{u^2 + 3} du = \lim_{t \to \infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Big|_3^{t-4}$

$b^2 - 4ac = 64 - 4(1)(19) \ominus$

Converges

$$= \lim_{t \to \infty} \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{t-4}{\sqrt{3}}\right) - \arctan\left(\frac{3}{\sqrt{3}}\right) \right] = \frac{1}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{3}\right] = \frac{1}{\sqrt{3}}\left[\frac{\pi}{6}\right] = \boxed{\frac{\pi}{6\sqrt{3}}}$$
$$\frac{3\pi}{6} \qquad \frac{2\pi}{6}$$

5. $\displaystyle \int_2^\infty \frac{1}{x^2 - 2x + 4} dx = \lim_{t \to \infty} \int_2^t \frac{1}{x^2 - 2x + 4} dx = \lim_{t \to \infty} \int_2^t \frac{1}{(x-1)^2 + 3} dx$

Complete Square $\quad x^2 - 2x + 1$

$\boxed{u = x - 1 \\ du = dx}$ $\boxed{x = 2 \Rightarrow u = 1 \\ x = t \Rightarrow u = t - 1}$ $\displaystyle = \lim_{t \to \infty} \int_1^{t-1} \frac{1}{u^2 + 3} du = \lim_{t \to \infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Big|_1^{t-1}$

$b^2 - 4ac = 4 - 4(1)(4) = \ominus$

Converges

$$= \lim_{t \to \infty} \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{t-1}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right) \right] = \frac{1}{\sqrt{3}}\left[\frac{2\pi}{6}\right] = \boxed{\frac{\pi}{3\sqrt{3}}}$$
$$\frac{\pi}{2} \qquad \frac{\pi}{6}$$

6. $\displaystyle \int \frac{1}{x^2 + 2x + 2} dx = \int \frac{1}{(x+1)^2 + 1} dx = \int \frac{1}{u^2 + 1} du = \arctan u + C = \boxed{\arctan(x+1) + C}$

Complete Square

$b^2 - 4ac = 4 - 4(1)(2) = \ominus$ $\quad x^2 + 2x + 1$

7. $\displaystyle \int_0^{\pi/2} \tan x \, dx = \lim_{t \to \pi/2^-} \int_0^t \underbrace{\frac{\sin x}{\cos x}}_{\tan x} dx = \lim_{t \to \pi/2^-} -\int_1^{\cos t} \frac{1}{u} du = \lim_{t \to \pi/2^-} -\ln|u| \Big|_1^{\cos t}$

$u = \cos x$ $x = 0 \Rightarrow u = \cos 0 = 1$

$du = -\sin x\, dx$ $x = t \Rightarrow u = \cos t$

$-du = \sin x\, dx$

Diverges

$$= \lim_{t \to \frac{\pi}{2}^-} -\ln|\cos t| + \ln 1 = -(-\infty) = \boxed{\infty}$$

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8. $\displaystyle\int_3^{\textcircled{4}} \frac{1}{(x-4)^2}\, dx = \lim_{t \to 4^-} \int_3^t \frac{1}{(x-4)^2}\, dx = \lim_{t \to 4^-} \int_{-1}^{t-4} \frac{1}{u^2}\, du = \lim_{t \to \infty} \left.\frac{-1}{u}\right|_{-1}^{t-4}$

$\boxed{\begin{array}{l} u = x-4 \\ du = dx \end{array}}$ $\boxed{\begin{array}{l} x = 3 \Rightarrow u = -1 \\ x = t \Rightarrow u = t-4 \end{array}}$

$$= \lim_{t \to 4^-} \frac{-1}{t-4} \mp \frac{1}{-1} = \boxed{\infty} \qquad \text{Diverges}$$

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9. $\displaystyle\int_{\textcircled{1}}^2 \frac{1}{x \ln x}\, dx = \lim_{t \to 1^+} \int_t^2 \frac{1}{x \ln x}\, dx = \lim_{t \to 1^+} \int_{\ln t}^{\ln 2} \frac{1}{u}\, du = \lim_{t \to 1^+} \left.\ln|u|\right|_{\ln t}^{\ln 2}$

$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\, dx \end{array}}$ $\boxed{\begin{array}{l} x = t \Rightarrow u = \ln t \\ x = 2 \Rightarrow u = \ln 2 \end{array}}$

$$= \lim_{t \to 1^+} \ln|\ln 2| - \ln|\ln t| = \boxed{\infty} \qquad \text{Diverges}$$

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10. $\displaystyle\int_{\textcircled{0}}^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = \lim_{t \to 0^+} \int_t^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = \lim_{t \to 0^+} 2\int_{\sqrt{t}}^1 e^u\, du = \lim_{t \to 0^+} \left.2e^u\right|_{\sqrt{t}}^1$

$\boxed{\begin{array}{l} u = \sqrt{x} \\ du = \frac{1}{2\sqrt{x}}\, dx \\ 2\,du = \frac{1}{\sqrt{x}}\, dx \end{array}}$ $\boxed{\begin{array}{l} x = t \Rightarrow u = \sqrt{t} \\ x = 1 \Rightarrow \sqrt{1} = 1 \end{array}}$

$$= \lim_{t \to 0^+} 2\left[e - e^{\sqrt{t}}\right] = \boxed{2e - 2} \qquad \text{Converges}$$

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11. $\displaystyle\int_1^\infty \frac{\ln x}{x}\, dx = \lim_{t \to \infty} \int_1^t \frac{\ln x}{x}\, dx = \lim_{t \to \infty} \int_0^{\ln t} u\, du = \lim_{t \to \infty} \left.\frac{u^2}{2}\right|_0^{\ln t}$

$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\, dx \end{array}}$ $\boxed{\begin{array}{l} x = 1 \Rightarrow u = \ln 1 = 0 \\ x = t \Rightarrow u = \ln t \end{array}}$

$$= \lim_{t \to \infty} \frac{(\ln t)^2}{2} - 0 = \boxed{\infty} \qquad \text{Diverges}$$

12. $\displaystyle\int \frac{2x-5}{x^2+2x+2}\,dx = \int \frac{2x-5}{(x+1)^2+1}\,dx = \int \frac{2(u-1)-5}{u^2+1}\,du = \int \frac{\overbrace{2u-2-5}^{}}{u^2+1}\,du$ Split · Split

$2u-7$

$b^2-4ac = 4-4(1)(2) = \ominus$

$\boxed{\begin{array}{l} u=x+1 \Rightarrow x=u-1 \\ du=dx \end{array}}$

$= 2\int \frac{u}{u^2+1} - 7\int \frac{1}{u^2+1}\,du$

$(x+1)^2 = x^2+2x+1$

$= \ln|u^2+1| - 7\arctan u + C$

$\boxed{= \ln\left|(x+1)^2+1\right| - 7\arctan(x+1) + c}$

$\underbrace{\qquad\qquad}_{x^2+2x+2}$

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13. $\displaystyle\int_{\textcircled{0}}^{1} \frac{e^x}{\sqrt{e^x-1}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{e^x}{\sqrt{e^x-1}}\,dx = \lim_{t\to 0^+}\int_{e^t-1}^{e-1} \frac{1}{\sqrt{u}}\,du = \lim_{t\to 0^+} 2\sqrt{u}\Big|_{e^t-1}^{e-1}$

$u^{-1/2}$

$\boxed{\begin{array}{l} u=e^x-1 \\ du=e^x dx \end{array}} \quad \boxed{\begin{array}{l} x=t \Rightarrow u=e^t-1 \\ x=1 \Rightarrow u=e-1 \end{array}}$

$= \lim_{t\to 0^+} 2\sqrt{e-1} - 2\sqrt{e^t-1} = \boxed{2\sqrt{e-1}}$ Converges

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14. $\displaystyle\int_{\textcircled{0}}^{1} \ln x\,dx = \lim_{t\to 0^+}\int_t^1 \ln x\cdot 1\,dx = \lim_{t\to 0^+} x\ln x\Big|_t^1 - \int_t^1 1\,dx$

$\boxed{\begin{array}{ll} u=\ln x & dv=1\,dx \\ du=\frac{1}{x}dx & v=x \end{array}}$

$= \lim_{t\to 0^+} x\ln x\Big|_t^1 - x\Big|_t^1 = \lim_{t\to 0^+} \ln 1 - t\ln t - (1-t) = \boxed{-1}$ Converges

$0\cdot(-\infty) \quad (*)$

$(*)\ \lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} \overset{\frac{-\infty}{\infty}}{\underset{L'H}{=}} \lim_{t\to 0^+} \frac{\frac{1}{t}(-t^2)}{-\frac{1}{t^2}} = \lim_{t\to 0^+} -t = 0$

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15. $\displaystyle\int_{\textcircled{0}}^{1} \frac{1}{(1-x^2)^{3/2}}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{(1-x^2)^{3/2}} = \lim_{t\to 1^-}\int_0^t \frac{1}{(\sqrt{1-x^2})^3}\,dx$

$\boxed{\begin{array}{l} x=\sin\theta \\ dx=\cos\theta\,d\theta \end{array}}$

$= \lim_{t\to 1^-}\int_{x=0}^{x=t} \frac{1}{(\sqrt{1-\sin^2\theta})^3}\cdot\cos\theta\,d\theta = \lim_{t\to 1^-}\int_{x=0}^{x=t} \frac{1}{\cos^2\theta}\,d\theta$

$$\left(\sqrt{\cos^2\theta}\right)^3 \longmapsto \cos^3\theta$$

$\Rightarrow \sqrt{1-x^2}$
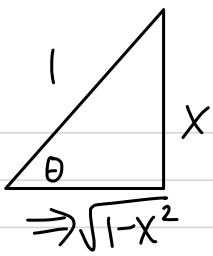
$$= \lim_{t\to 1^-} \int_{x=0}^{x=t} \sec^2\theta\, d\theta = \lim_{t\to 1^-} \tan\theta \Big|_{x=0}^{x=t}$$

$$= \lim_{t\to 1^-} \frac{x}{\sqrt{1-x^2}}\Big|_0^t = \lim_{t\to 1^-} \frac{t}{\sqrt{1-t^2}} - 0 = \boxed{\infty} \quad \text{Diverges}$$

16.
$$\int_{-\infty}^{3} \frac{1}{x^2-6x+10}\, dx = \lim_{s\to -\infty} \int_{s}^{3} \frac{1}{x^2-6x+10}\, dx$$

$b^2-4ac = 36 - 4(1)(10)$

$$\overset{\text{Complete Sq}}{=} \lim_{s\to -\infty} \int_{s}^{3} \frac{1}{(x-3)^2+1}\, dx$$

$u = x-3$
$du = dx$

$x = s \Rightarrow u = s-3$
$x = 3 \Rightarrow u = 0$
$x = t \Rightarrow u = t-3$

$$= \lim_{s\to -\infty} \int_{s-3}^{0} \frac{1}{u^2+1}$$

$$= \lim_{s\to -\infty} \arctan u \Big|_{s-3}^{0}$$

$$= \lim_{s\to -\infty} \arctan 0 - \arctan(s-3)$$

$$= \boxed{\frac{\pi}{2}} \qquad \text{Integral Converges.}$$

17. $\displaystyle\int_{-5}^{0} \frac{x}{x^2+4x-5} dx = \int_{-5}^{0} \frac{x}{(x+5)(x-1)} dx = \lim_{t\to-5^+} \int_{t}^{0} \frac{x}{(x+5)(x-1)} dx$

$x=-5$

$\displaystyle = \lim_{t\to-5^+} \int_{t}^{0} \frac{\frac{5}{6}}{(x+5)} + \frac{\frac{1}{6}}{x-1} dx$

$\displaystyle = \lim_{t\to-5^+} \frac{5}{6}\ln|x+5| + \frac{1}{6}\ln|x-1| \Big|_{t}^{0}$

$\displaystyle = \lim_{t\to-5^+} \frac{5}{6}\ln 5 - \frac{1}{6}\ln 1 \ominus \left(\frac{5}{6}\ln|t+5| + \frac{1}{6}\ln|t-1|\right)$

0   0^+ $\ln 6$

$-\infty$

$= \boxed{\infty}$ Diverges

PFD

$(x+5)(x-1)\left[\dfrac{x}{(x+5)(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{x-1}\right](x+5)(x-1)$

$x = A(x-1) + B(x+5)$

$= Ax - A + Bx + 5B$

$= (A+B)x + 5B - A$

Conditions

· $A + B = 1 \implies B = 1-A$
· $5B - A = 0$

$5(1-A) - A = 0$

$5 - 5A - A = 0$

$6A = 5$

$A = \frac{5}{6} \implies B = 1 - \frac{5}{6} = \frac{1}{6}$

18. $\displaystyle\int_0^2 \frac{1}{(x-2)^2}\,dx = \int_0^{\boxed{2}} \frac{1}{(x-2)^2}\,dx = \lim_{s\to 2^-}\int_0^s \frac{1}{(x-2)^2}\,dx = \lim_{s\to 2^-} \left.\frac{-1}{x-2}\right|_0^s$

$$= \lim_{s\to 2^-} \underbrace{-\frac{1}{s-2}}_{\;} + \frac{1}{-2}$$

↳ u-sub?!

$$+\infty$$

Diverges

19. $\displaystyle\int_0^{\boxed{\infty}} \frac{1}{x^2+3x+2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{x^2+3x+2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{(x+2)(x+1)}\,dx$

PFD

$$\frac{1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$(x+2)(x+1)\left[\;\right]$$

$$1 = A(x+1) + B(x+2)$$
$$= Ax + A + Bx + 2B$$
$$= (A+B)x + A + 2B$$

Conditions

- $A+B=0 \Rightarrow A=-B$
- $A+2B=1 \quad -B+2B=1$
$$B=1 \Rightarrow A=-1$$

$$= \lim_{t\to\infty}\int_0^t \frac{-1}{x+2} + \frac{1}{x+1}\,dx$$

$$= \lim_{t\to\infty} \left.-\ln|x+2| + \ln|x+1|\right|_0^t$$

$$= \lim_{t\to\infty} \underbrace{-\ln|t+2| + \ln|t+1|}_{\text{Indeterminate Difference}} - \left(-\ln 2 + \ln 1\right)$$

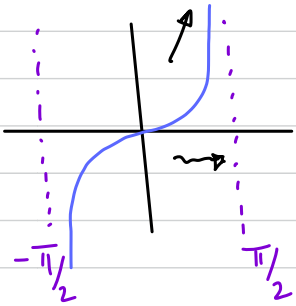$$= \lim_{t\to\infty} \ln\left|\frac{t+1}{t+2}\,\frac{1/t}{1/t}\right| + \ln 2$$

$$= \lim_{t\to\infty} \ln\left|\frac{1+\frac{1}{t}}{1+\frac{2}{t}}\right| + \ln 2 = \boxed{\ln 2}$$

Converges

20. $\displaystyle\int_0^{\pi/2} \tan^2 x\, dx = \lim_{t\to \pi/2^-} \int_0^t \tan^2 x\, dx = \lim_{t\to\pi/2^-} \int_0^t \sec^2 x - 1\, dx$

$$= \lim_{t\to\pi/2^-} \tan x - x \Big|_0^t = \lim_{t\to\pi/2^-} \tan t - t - (\tan 0 - 0)$$

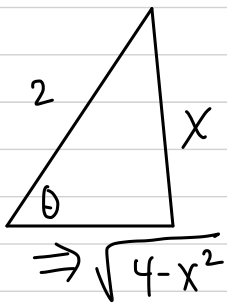(with annotations: $\tan t \to +\infty$, $t \to \pi/2$, $\tan 0 \to 0$)

$$= \infty \qquad \text{Diverges}$$

21. $\displaystyle\int_0^2 \frac{1}{(4-x^2)^{3/2}}\, dx = \lim_{t\to 2^-} \int_0^t \frac{1}{\left(\sqrt{4-x^2}\right)^3}\, dx = \lim_{t\to 2^-}\int_{x=0}^{x=t} \frac{1}{\left(\sqrt{4-4\sin^2\theta}\right)^3}\, 2\cos\theta\, d\theta$

$$\sqrt{4(1-\sin^2\theta)}$$
$$\sqrt{4\cos^2\theta} \to (2\cos\theta)^3$$

$$\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{array}}$$

$\Rightarrow \sqrt{4-x^2}$

$$= \lim_{t\to 2^-} \frac{1}{4}\int_{x=0}^{x=t} \frac{1}{\cos^2\theta}\, d\theta = \lim_{t\to 2^-} \frac{1}{4}\int_{x=0}^{x=t} \sec^2\theta\, d\theta$$

$$= \lim_{t\to 2^-} \frac{1}{4}\tan\theta \Big|_{x=0}^{x=t} = \lim_{t\to 2^-} \frac{1}{4}\left(\frac{x}{\sqrt{4-x^2}}\right)\Big|_{x=0}^{x=t}$$

$$= \lim_{t\to 2^-} \frac{1}{4}\left(\frac{t}{\sqrt{4-t^2}}\right) - \frac{1}{4}\cdot 0 = \infty \qquad \text{Diverges}$$

(with annotations: $t \to 2$, $\sqrt{4-t^2} \to 0^+$, $\to +\infty$, 0^+)

22. $\displaystyle\int \frac{4x^2+7x+6}{(x+2)(x^2+4)}\,dx = \int \frac{1}{x+2} + \frac{3x+1}{x^2+4}\,dx = \int \frac{1}{x+2} + \frac{3x}{x^2+4} + \frac{1}{x^2+4}\,dx$

$$= \ln|x+2| + \frac{3}{2}\ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

PFD

$(x+2)\overline{(x^2+4)\left[\dfrac{4x^2+7x+6}{(x+2)(x^2+4)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+4}\right](x+2)(x^2+4)}$

$$4x^2+7x+6 = A(x^2+4)+(Bx+C)(x+2)$$
$$= Ax^2+4A+Bx^2+2Bx+Cx+2C$$
$$= (A+B)x^2 + (2B+C)x + 4A+2C$$

Conditions

- $A+B=4 \Rightarrow B=4-A$
- $2B+C=7 \quad 2(4-A)+C=7 \Rightarrow 8-2A+C=7$
- $4A+2C=6 \qquad\qquad\qquad C=2A-1$

$$4A+2(2A-1)=6$$
$$4A+4A-2=6$$
$$8A=8$$

$C=2-1=1 \Longleftarrow A=1 \Rightarrow B=4-1=3$

23. $\displaystyle\int_1^{\infty} \frac{1}{x(x+1)}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x(x+1)}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x} - \frac{1}{x+1}\,dx$

PFD

$x(x+1)\overline{\left[\dfrac{1}{x(x+1)} = \dfrac{A}{x} + \dfrac{B}{x+1}\right]x(x+1)}$

$$1 = A(x+1)+Bx$$
$$= (A+B)x + A$$

Conditions
- $A+B=0$
- $A=1 \Rightarrow B=-1$

☺

$$= \lim_{t\to\infty} \ln|x| - \ln|x+1| \Big|_1^t$$

$$= \lim_{t\to\infty} \underbrace{\ln|t| - \ln|t+1| - (\ln 1 - \ln 2)}_{\text{Indeterminate Difference}}$$

$$= \lim_{t\to\infty} \ln\left|\frac{t}{t+1}\,\frac{1/t}{1/t}\right| + \ln 2$$

$\hookrightarrow$ OR L'H

0

$$= \lim_{t \to \infty} \ln\left|\frac{1}{1+\frac{1}{t}}\right|_0 + \ln 2 = \boxed{\ln 2} \quad \text{Converges}$$

24. $\displaystyle\int_{-3}^{-2} \frac{1}{x^2-4} dx = \int_{-3}^{\boxed{-2}} \frac{1}{(x-2)(x+2)} dx = \lim_{t \to -2^-} \int_{-3}^{t} \frac{1}{(x-2)(x+2)} dx$

$x = -2$

$\underline{\text{PFD}}$

$$(x-2)(x+2)\left[\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}\right](x-2)(x+2)$$

$$1 = A(x+2) + B(x-2)$$

$$= Ax + 2A + Bx - 2B$$

$$= (A+B)x + 2A - 2B$$

Conditions
- $A + B = 0 \Rightarrow B = -A$
- $2A - 2B = 1$

$$2A - 2(-A) = 1$$
$$4A = 1$$
$$A = \tfrac{1}{4} \Rightarrow B = -\tfrac{1}{4}$$

$$= \lim_{t \to -2^-} \int_{3}^{t} \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2} dx$$

$$= \lim_{t \to -2^-} \tfrac{1}{4}\ln|x-2| - \tfrac{1}{4}\ln|x+2| \Big|_{3}^{t}$$

$$= \lim_{t \to -2^-} \tfrac{1}{4}\ln|t-2| - \tfrac{1}{4}\ln|t+2| - \left(\tfrac{1}{4}\ln| \, | - \tfrac{1}{4}\ln 5\right)$$

$$= \boxed{\infty} \quad \text{Diverges}$$

25. $\displaystyle\int_{0}^{1} \arcsin x \, dx = \int_{0}^{1} \arcsin x \cdot 1 \, dx = x\arcsin x \Big|_{0}^{1} - \int_{0}^{\boxed{1}} \frac{x}{\sqrt{1-x^2}} dx$

$\underline{\text{IBP}}$

$u = \arcsin x$	$dv = 1\,dx$
$du = \dfrac{1}{\sqrt{1-x^2}} dx$	$v = x$

$u = 1-x^2$ $x = 0 \Rightarrow u = 1$
$du = -2x\,dx$ $x = t \Rightarrow u = 1-t^2$
$-\tfrac{1}{2} du = x\,dx$

$$= x\arcsin x \Big|_{0}^{1} - \lim_{t \to 1^-} \int_{0}^{t} \frac{x}{\sqrt{1-x^2}} dx$$

$$= x\arcsin x \Big|_{0}^{1} + \lim_{t \to 1^-} \tfrac{1}{2}\int_{1}^{1-t^2} \frac{1}{\sqrt{u}} du$$

$$= x\arcsin x \Big|_{0}^{1} + \lim_{t \to 1^-} \sqrt{u} \Big|_{1}^{1-t^2}$$

$$= \arcsin 1 - 0 + \lim_{t \to 1^-} \sqrt{1-t^2} - \sqrt{1} = \boxed{\tfrac{\pi}{2} - 1} \quad \text{Converges}$$

26. $\displaystyle\int_0^1 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0^+} -\int_{1/t}^1 e^u\,du$

$$u = 1/x$$
$$du = -\frac{1}{x^2}\,dx$$
$$-du = \frac{1}{x^2}\,dx$$

$$x = t \Rightarrow u = 1/t$$
$$x = 1 \Rightarrow u = 1$$

$= \lim_{t\to 0^+} -e^u \Big|_{1/t}^1 = \lim_{t\to 0^+} -e + e^{1/t}$

$= \boxed{\infty}$ Diverges

Same u-sub

27. $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0^-}\int_{-1}^t \frac{e^{1/x}}{x^2} \overset{!}{=} \ldots = \lim_{t\to 0^-} -e^{1/x}\Big|_{-1}^t$

$= \lim_{t\to 0^-} -e^{1/t} + e^{-1} = \boxed{\dfrac{1}{e}}$ Converges

28. $\displaystyle\int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx = \lim_{s\to -\infty}\int_s^0 \frac{x^2}{9+x^6}\,dx = \lim_{s\to -\infty}\frac{1}{3}\int_{s^3}^0 \frac{1}{9+u^2}\,du$

$(x^3)^2$

$$u = x^3$$
$$du = 3x^2\,dx$$
$$\frac{1}{3}\,du = x^2\,dx$$

$$x = s \Rightarrow u = s^3$$
$$x = 0 \Rightarrow u = 0$$

$= \lim_{s\to -\infty} \frac{1}{3}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right)\Big|_{s^3}^0$

$= \lim_{s\to -\infty} \frac{1}{9}\left[\arctan 0 - \arctan\left(\frac{s^3}{3}\right)\right]$

$= \frac{1}{9}\left(\frac{\pi}{2}\right) = \boxed{\dfrac{\pi}{18}}$ Converges

29. $\displaystyle\int_2^{\infty} \frac{x}{e^{3x}}\, dx = \lim_{t\to\infty} \int_2^t x e^{-3x}\, dx = \lim_{t\to\infty} \frac{-x}{3e^{3x}}\Big|_2^t + \frac{1}{3}\int_2^t e^{-3x}\, dx$

$\searrow \dfrac{e^{-3x}}{-3}$

IBP

$u = x$	$dv = e^{-3x}\, dx$
$du = dx$	$v = \dfrac{e^{-3x}}{-3}$

$= \displaystyle\lim_{t\to\infty} \frac{-x}{3e^{3x}}\Big|_2^t - \frac{1}{9e^{3x}}\Big|_2^t$

$= \displaystyle\lim_{t\to\infty} \underbrace{\frac{-t}{3e^{3t}}}_{-\infty/\infty} + \frac{2}{3e^6} - \underbrace{\frac{1}{9e^{3t}}}_{\to 0\;\nearrow\;\infty} + \frac{1}{9e^6}$

$\underset{L'H}{=} \displaystyle\lim_{t\to\infty} \underbrace{\frac{-1}{9e^{3t}}}_{\to 0 \;/\; \infty} + \frac{6}{9e^6} + \frac{1}{9e^6} = \boxed{\dfrac{7}{9e^6}}$ Converges

---

30. $\displaystyle\int_0^e \frac{\ln x}{\sqrt{x}}\, dx = \lim_{t\to 0^+} \int_t^e \ln x \cdot x^{-1/2}\, dx = \lim_{t\to 0^+} 2\sqrt{x}\,\ln x\Big|_t^e - 2\int_t^e \frac{1}{\sqrt{x}}\, dx$

$\nearrow x^{-1/2}$

IBP

$u = \ln x$	$dv = x^{-1/2}\, dx$
$du = \dfrac{1}{x}\, dx$	$v = 2\sqrt{x}$

$= \displaystyle\lim_{t\to 0^+} 2\sqrt{x}\,\ln x\Big|_t^e - 4\sqrt{x}\Big|_t^e$

$= \displaystyle\lim_{t\to 0^+} \underbrace{2\sqrt{e}\,\ln e}_{1} \underbrace{- 2\sqrt{t}\,\ln t}_{\substack{0\cdot(-\infty) \\ (\ast)\;\to 0}} - 4\sqrt{e} + \underbrace{4\sqrt{t}}_{\to 0}$

$= 2\sqrt{e} - 4\sqrt{e} = \boxed{-2\sqrt{e}}$ Converges

$(\ast)$ $\displaystyle\lim_{t\to 0^+} \overset{0\cdot(-\infty)}{\sqrt{t}\cdot\ln t} \underset{L'H}{=} \lim_{t\to 0^+} \frac{\overset{-\infty/\infty}{\ln t}}{\frac{1}{\sqrt{t}}} = \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-1}{2t^{3/2}}} \underset{-2t^{3/2}}{} = \lim_{t\to 0^+} \overset{0}{-2\sqrt{t}} = 0$

$t^{-1/2} \longrightarrow -\frac{1}{2}t^{-3/2}$

Sequences

simplify $\frac{n}{n^4}$ **or indeterminate** $\frac{\infty}{\infty}$

31. $\lim\limits_{n\to\infty} \dfrac{1+n-7n^4}{3n^4+8n^3+9} \cdot \dfrac{1/n^4}{1/n^4} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n^4}{}^{\to 0}+\frac{1}{n^3}{}^{\to 0}-7}{3+\frac{8}{n}+\frac{9}{n^4}{}_{\to 0}} = \boxed{\dfrac{-7}{3}}$ Converges

32. $\lim\limits_{n\to\infty} \dfrac{n^3}{(n+1)^3} {}^{\frac{\infty}{\infty}} = \lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\cdot\dfrac{1/n}{1/n}\right)^3 = \lim\limits_{n\to\infty}\left(\dfrac{1}{1+\frac{1}{n}}\right)^3 = \boxed{1}$ Converges

33. $\lim\limits_{n\to\infty}\left(\dfrac{n-5}{n}\right)^n = \lim\limits_{n\to\infty}\left(1-\dfrac{5}{n}\right)^n {}^{1^\infty} = \lim\limits_{x\to\infty}\left(1-\dfrac{5}{x}\right)^x = e^{\lim\limits_{x\to\infty} \ln\left[\left(1-\frac{5}{x}\right)^x\right]}$

$= e^{\lim\limits_{x\to\infty} x^{\infty}\cdot\ln\left(1-\frac{5}{x}\right)^{\to 0}} = e^{\lim\limits_{x\to\infty}\frac{\ln\left(1-\frac{5}{x}\right)}{\frac{1}{x}}{}^{\frac{0}{0}}}$ Converges

$\underset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\dfrac{\frac{1}{1-\frac{5}{x}}\left(\frac{5}{x^2}\right)}{-\frac{1}{x^2}}} = e^{\lim\limits_{x\to\infty}\frac{1}{1-\frac{5}{x}}(-5)} = \boxed{e^{-5}}$

34. $\lim\limits_{n\to\infty}\dfrac{(2n+3)!}{(2n+5)!} = \lim\limits_{n\to\infty}\dfrac{(2n+3)!}{(2n+5)(2n+4)(2n+3)!} = \lim\limits_{n\to\infty}\dfrac{1}{(2n+5)(2n+4)}{}^{\to 0}$

$= \boxed{0}$ Converges

OR expand

$= \lim\limits_{n\to\infty}\dfrac{(2n+3)(2n+2)(2n+1)(2n)\cdots 5\cdot4\cdot3\cdot2\cdot1}{(2n+5)(2n+4)(2n+3)(2n+2)\cdots 5\cdot4\cdot3\cdot2\cdot1} = \boxed{0}$ same

35.

$\lim\limits_{n\to\infty}\arctan(n^2+1)^{\to\infty} = \boxed{\dfrac{\pi}{2}}$ Converges

36. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{(\ln n)^2} \overset{\infty/\infty}{=} \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{(\ln x)^2} \underset{L'H}{\overset{\infty/\infty}{=}} \lim\limits_{x\to\infty} \dfrac{\frac{1}{2\sqrt{x}}}{2(\ln x)\cdot\frac{1}{x}}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{4\ln x} \underset{L'H}{\overset{\infty/\infty}{=}} \lim\limits_{x\to\infty} \dfrac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}} = \lim\limits_{x\to\infty} \dfrac{\sqrt{x}}{8} = \boxed{\infty}$

Diverges

37. $\overset{\infty^0}{} \quad \lim\limits_{n\to\infty} n^{1/n} = \lim\limits_{x\to\infty} x^{1/x} = e^{\lim\limits_{x\to\infty} \ln\left[x^{1/x}\right]} = e^{\lim\limits_{x\to\infty} \frac{1}{x}\cdot \ln x}$

$= e^{\lim\limits_{x\to\infty} \frac{\ln x}{x}} \underset{L'H}{\overset{\infty/\infty}{=}} e^{\lim\limits_{x\to\infty} \frac{1/x}{1}} = e^0 = \boxed{1}$ Converges

38. $\overset{\infty\cdot 0}{} \quad \lim\limits_{n\to\infty} n\cdot \sin\left(\tfrac{1}{n}\right) = \lim\limits_{x\to\infty} x\cdot \sin\left(\tfrac{1}{x}\right) = \lim\limits_{x\to\infty} \dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{0/0}{}$

$= \lim\limits_{x\to\infty} \dfrac{\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \cos\left(\tfrac{1}{x}\right) = \boxed{1}$ Converges

39. $\overset{1^\infty}{} \quad \lim\limits_{n\to\infty} \left(\dfrac{n+1}{n}\right)^n = \lim\limits_{x\to\infty}\left(\dfrac{x+1}{x}\right)^x = \lim\limits_{x\to\infty}\left(1+\tfrac{1}{x}\right)^x = e^{\lim\limits_{x\to\infty} \ln\left[\left(1+\frac{1}{x}\right)^x\right]}$

$= e^{\lim\limits_{x\to\infty} x\cdot \ln\left(1+\frac{1}{x}\right)} = e^{\lim\limits_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \overset{0/0}{}}$

$\underset{L'H}{=} e^{\lim\limits_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}} = \boxed{e}$ Learn for Ratio Test.

40. $\lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^n \overset{1^\infty}{=} \lim\limits_{x\to\infty}\left(\dfrac{x}{x+1}\right)^x = e^{\lim\limits_{x\to\infty}\ln\left[\left(\frac{x}{x+1}\right)^x\right]}$

$= e^{\lim\limits_{x\to\infty}\overset{\infty\cdot 0}{x}\ln\left(\frac{x}{x+1}\right)} = e^{\lim\limits_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}}\,\%_0}$

$\overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\dfrac{\frac{1}{\left(\frac{x}{x+1}\right)}\left[\frac{(x+1)(1)-x(1)}{(x+1)^2}\right]}{-\frac{1}{x^2}}}$ $\quad -x^2$

$= e^{\lim\limits_{x\to\infty}\dfrac{x+1}{x}\left[\frac{1}{(x+1)^2}\right]\cdot(-x^2)}$ $\qquad$ Converges

$= e^{\lim\limits_{x\to\infty}\frac{-x}{x+1}}\overset{-\infty/\infty}{\underset{\text{L'H}}{=}} e^{\lim\limits_{x\to\infty}\frac{-1}{1}} = e^{-1} = \boxed{\dfrac{1}{e}}$

$\text{or }\dfrac{\frac{1}{x}}{\frac{1}{x}}$

Series Sums

41. $\displaystyle\sum_{n=1}^{\infty}\dfrac{2^n+3^n}{6^n} = \sum_{n=1}^{\infty}\dfrac{2^n}{6^n}+\sum_{n=1}^{\infty}\dfrac{3^n}{6^n} = \sum_{n=1}^{\infty}\left(\dfrac{2}{6}\right)^n+\sum_{n=1}^{\infty}\left(\dfrac{3}{6}\right)^n$

$= \displaystyle\sum_{n=1}^{\infty}\dfrac{1}{3^n}+\sum_{n=1}^{\infty}\dfrac{1}{2^n}$

$a=\frac{1}{3}$ $\qquad = \dfrac{1}{3}+\dfrac{1}{3^2}+\dfrac{1}{3^3}+\cdots$ $\qquad = \dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\cdots$ $\qquad a=\frac{1}{2}$

$r=\frac{1}{3}$ $\qquad$ Conv. GST $|r|=\frac{1}{3}<1$ $\qquad$ Conv. GST $|r|=\frac{1}{2}<1$ $\qquad r=\frac{1}{2}$

$\text{Sum}=\dfrac{a}{1-r}=\dfrac{\frac{1}{3}}{1-\frac{1}{3}}=\dfrac{\frac{1}{3}}{\frac{2}{3}}=\dfrac{1}{2}$ $\quad + \quad$ $\text{Sum}=\dfrac{a}{1-r}=\dfrac{\frac{1}{2}}{1-\frac{1}{2}}=\dfrac{\frac{1}{2}}{\frac{1}{2}}=1$

$$\frac{1}{2}+1=\boxed{\frac{3}{2}}$$

Sum of 2 Convergent Series
<u>is</u> Convergent

not needed here b/c didn't ask to <u>prove</u> Converge

42. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,2^{n-1}}{3^{n+1}} = \underbrace{\frac{2^0}{3^2}}_{a} - \frac{2^1}{3^3} + \frac{2^2}{3^4}+\dots$

n=1   n=2   n=3

$r=\dfrac{2}{3}$

$a=\dfrac{1}{9}$

$r=\dfrac{-2}{3}$

$\left[\text{Convergent GST } |r|=\left|-\tfrac{2}{3}\right|=\tfrac{2}{3}<1\right]$ ← not needed for Just Sum

$$\text{Sum}=\frac{a}{1-r}=\frac{\frac{1}{9}}{1-\left(-\frac{2}{3}\right)}=\frac{\frac{1}{9}}{\frac{5}{3}} \;\overset{3}{\underset{5}{}}\; =\boxed{\frac{1}{15}}$$

43. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\,3^{n+2}}{2^{4n-1}} = \underbrace{\frac{-3^3}{2^3}}_{a} + \frac{3^4}{2^7} - \frac{3^5}{2^{11}} + \dots$

n=1   n=2   n=3

$r=-\dfrac{3}{2^4}$

$a=\dfrac{-3^3}{2^3}=\dfrac{-27}{8}$

$r=-\dfrac{3}{2^4}=\dfrac{-3}{16}$

not needed for Just Sum

Convergent GST $|r|=\left|-\tfrac{3}{16}\right|=\tfrac{3}{16}<1$

$$\text{Sum}=\frac{a}{1-r}=\frac{\frac{-27}{8}}{1-\left(-\frac{3}{16}\right)}=\frac{\frac{-27}{8}}{\frac{19}{16}}\;\overset{2}{\underset{}{\tfrac{16}{19}}}\; =\boxed{\frac{-54}{19}}$$

44. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\,4^n}{3^{2n-1}} = \underbrace{\frac{-4}{3^1}}_{a} + \frac{4^2}{3^3} - \frac{4^3}{3^5} + \dots$

n=1   n=2   n=3

$r=\dfrac{-4}{3^2}$

$a=\dfrac{-4}{3}$

$r=\dfrac{-4}{3^2}=\dfrac{-4}{9}$

not needed for Just Sum

Convergent GST $|r|=\left|-\tfrac{4}{9}\right|=\tfrac{4}{9}<1$

$$\text{Sum}=\frac{a}{1-r}=\frac{\frac{-4}{3}}{1-\left(-\frac{4}{9}\right)}=\frac{\frac{-4}{3}}{\frac{13}{9}}\;\overset{3}{\underset{}{\tfrac{9}{13}}}\; =\boxed{\frac{-12}{13}}$$

More Series

45. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^3 - \ln n}{5n^3 - 9}$ $\longrightarrow$ $\displaystyle\lim_{n\to\infty} \frac{2n^3 - \ln n}{5n^3 - 9} \cdot \frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{2 - \frac{\ln n}{n^3}\nearrow 0}{5 - \frac{9}{n^3}\searrow 0} = \frac{2}{5} \neq 0$

$\Rightarrow$ Series $\boxed{\text{Diverges}}$ by nTDT

See (☆)

$(\ast)$ $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^3} \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{\frac{1}{x}}{3x^2} \cdot \frac{1/3x^2}{} = \lim_{x\to\infty} \frac{1}{3x^3} \nearrow 0 = 0$

46. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ $\xrightarrow{\text{I.T.}}$ $f(x) = \dfrac{\ln x}{x^2}$

Pre Conditions

Positive $x > 1$

Continuous $x > 0$

Decreasing

$f'(x) = \dfrac{x^2(\frac{1}{x}) - \ln x(2x)}{x^4}$

$= \dfrac{x - 2x\ln x}{x^4}$

$= \dfrac{x(1 - 2\ln x)}{x^4}$

$= \dfrac{1 - 2\ln x}{x^3} < 0$ when

$1 - 2\ln x < 0$
$1 < 2\ln x$
$\dfrac{1}{2} < \ln x$
$e^2 \quad e$
$x > \sqrt{e}$ o.k.

Compute Integral

$\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \ln x \cdot x^{-2}\, dx$

$\boxed{\begin{array}{ll} u = \ln x & dv = x^{-2}dx \\ du = \frac{1}{x}dx & v = \frac{x^{-1}}{-1} = -\frac{1}{x} \end{array}}$

$= \lim_{t\to\infty} \left. \frac{-\ln x}{x}\right|_1^t + \int_1^t \frac{1}{x^2}\, dx \quad (x^{-2})$

$= \lim_{t\to\infty} \left. \frac{-\ln x}{x}\right|_1^t - \left. \frac{1}{x}\right|_1^t$

$= \lim_{t\to\infty} \frac{-\ln t}{t} \frac{-\infty}{} + \frac{\ln 1}{1}\nearrow 0 - \frac{1}{t}\nearrow 0 + 1$

$\underset{L'H}{=} \lim_{t\to\infty} \frac{-\frac{1}{t}\nearrow 0}{1} + 1 = \boxed{1}$ Integral Converges

$\Rightarrow$ Series $\boxed{\text{Converges}}$ by I.T.

$\boxed{\begin{array}{c}\text{Note: could also run CT if use} \\ \text{Bound } \ln n \leq \sqrt{n} \,..\end{array}}$

47. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{4n^2-2} \sim \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ Converges   p-Series $p=\frac{3}{2}>1$

$$\lim_{n\to\infty} \frac{\dfrac{\sqrt{n}+3}{4n^2-2}}{\dfrac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^2+3n^{3/2}}{4n^2-2} \cdot \frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{1+\dfrac{3}{\sqrt{n}}{\to 0}}{4-\dfrac{2}{n^2}{\to 0}} = \frac{1}{4}$$

Finite, Non-Zero

$\Rightarrow$ O.S. also $\boxed{\text{Converges}}$ by LCT

48. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{19}+40n^6+4n^3+19}{4+17n^5+n^{20}} \sim \sum_{n=1}^{\infty} \frac{n^{19}}{n^{20}} = \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges Harmonic p-Series $p=1$

$$\lim_{n\to\infty} \frac{\dfrac{n^{19}+40n^6+4n^3+19}{4+17n^5+n^{20}}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n^{20}+40n^7+4n^4+19n}{4+17n^5+n^{20}} \cdot \frac{1/n^{20}}{1/n^{20}}$$

$$= \lim_{n\to\infty} \frac{1+\dfrac{40}{n^{13}}{\to 0}+\dfrac{4}{n^{16}}{\to 0}+\dfrac{19}{n^{19}}{\to 0}}{\dfrac{4}{n^{20}}{\to 0}+\dfrac{17}{n^{15}}{\to 0}+1} = 1$$ Finite, Non-Zero

$\Rightarrow$ O.S. also $\boxed{\text{Diverges}}$ by LCT

49. $\displaystyle\sum_{n=2}^{\infty} \frac{e^n}{\ln n}$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{\ln n} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{e^x}{\ln x} \underset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{1/x}$$

$$= \lim_{x\to\infty} xe^x = \infty \neq 0$$

50. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^5} + \frac{1}{5^n} = \sum_{n=1}^{\infty} \frac{5}{n^5} + \sum_{n=1}^{\infty} \frac{1}{5^n}$

Constant Multiple of
Convergent p-Series
$p = 5 > 1$ $\underline{is}$
Convergent

Convergent Geometric
$|r| = \frac{1}{5} < 1$

$\not\! f$ not needed
{by Arithmetic of Series}
$\not\! f$

Sum of 2 Convergent Series $\underline{is}$ $\boxed{\text{Convergent}}$

51. $\displaystyle\sum_{n=1}^{\infty} \frac{1 + 3n^3}{n^5} \sim \sum_{n=1}^{\infty} \frac{n^3}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ Convergent p-Series $p = 2 > 1$

[Note: CT not helpful → bound in wrong order]

$$\lim_{n \to \infty} \frac{\frac{1 + 3n^3}{n^5}}{\frac{1}{n^2}} \cdot n^2 = \lim_{n \to \infty} \frac{n^2 + 3n^5}{n^5} = \lim_{n \to \infty} \frac{1}{n^3}^6 + 3 = 3$$

or $\dfrac{\frac{1}{n^5}}{\frac{1}{n^5}}$

Finite, Non-zero

$\Rightarrow$ O.S. also $\boxed{\text{Converges}}$ by LCT

positive $x > 1$
Continuous $x > 0$

52. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^7} \rightsquigarrow f(x) = \frac{1}{x(\ln x)^7} = \left[x(\ln x)^7\right]^{-1}$ Decreasing when

$$f'(x) = -\left[x(\ln x)^7\right]^{-2}\left[x \cdot 7(\ln x)^6 \cdot \frac{1}{x} - (\ln x)^7 (1)\right]$$

$$= \frac{-(\ln x)^6 \left[7 - \ln x\right]}{\left[x(\ln x)^7\right]^2} = \frac{-7 + \ln x}{x^2 (\ln x)^8} < 0$$

$\oplus \quad \oplus$

Compute

$$\int_2^{\infty} \frac{1}{x(\ln x)^7} dx = \lim_{t \to \infty} \int_2^t \frac{1}{x(\ln x)^7} dx$$

$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array}}$ $= \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{1}{u^7} du$

$\boxed{\begin{array}{l} x = 2 \Rightarrow u = \ln 2 \\ x = t \Rightarrow u = \ln t \end{array}}$ $= \lim_{t \to \infty} \frac{-1}{6u^6}\Big|_{\ln 2}^{\ln t}$

when $-7 + \ln x < 0 \Rightarrow \ln x > 7$
$\Rightarrow x > e^7$ ✓o.k.

$$= \lim_{t \to \infty} -\frac{1}{6(\ln t)^6} \nearrow^{0} \xcancel{} + \frac{1}{6(\ln 2)^6} = \frac{1}{6(\ln 2)^6} \qquad \text{Integral Converges}$$

$$\Rightarrow \text{ o.s. } \boxed{\text{Converges}} \text{ by I.T.}$$

53. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2} \sim \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ Constant Multiple of Convergent

$$p\text{-Series } p = 2 > 1 \text{ is } \text{Convergent}$$

Bound Terms

$$\frac{\arctan n}{1+n^2} \leq \frac{\frac{\pi}{2}}{1+n^2} \leq \frac{\frac{\pi}{2}}{n^2} \quad \text{and} \quad \Rightarrow \text{o.s. } \boxed{\text{Converges}} \text{ by CT}$$

OR

LCT

$$\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2} \sim \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{Conv. } p\text{-Series } p = 2 > 1$$

$$\lim_{n \to \infty} \frac{\frac{\arctan n}{1+n^2}}{\frac{1}{n^2}} = \lim_{n \to \infty} \arctan n \cdot \frac{n^2}{1+n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{\pi}{2} \lim_{n \to \infty} \frac{1}{1+\frac{1}{n^2}} \nearrow^{1} = \frac{\pi}{2}$$

$$\text{Finite, Non-Zero}$$

$$\Rightarrow \text{ o.s. } \boxed{\text{Converges}} \text{ by LCT}$$

OR Integral Test...

54. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{\arctan n}$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2+1}{\arctan n} \nearrow^{\infty} = \infty \neq 0.$$

$$\searrow_{\frac{\pi}{2}}$$

55. $\displaystyle\sum_{n=1}^{\infty} \frac{2\text{\textcircled{n}}+5}{5n^3+3n^2} \sim \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges, p-Series $p=2>1$

$$\lim_{n\to\infty} \frac{\dfrac{2n+5}{5n^3+3n^2}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^3+5n^2}{5n^3+3n^2}\cdot\frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{2+\cancel{5/n}^{\,0}}{5+\cancel{3/n}_{\,0}} = \frac{2}{5}$$

Finite

Non-Zero

$\Rightarrow$ O.S. also $\boxed{\text{Converges}}$ by LCT

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56. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+5n-3} \xrightarrow[\text{"Helpful"}]{\text{A.S.}} \sum_{n=1}^{\infty} \frac{1}{n^2+5n-3} \sim \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges, p-Series

$p=2>1$

work around

Bound Terms

$$\frac{1}{n^2+5n-3} \le \frac{1}{n^2} \quad \text{and} \quad$$

$\Rightarrow$ A.S. also Converges by C.T.

O.S. $\boxed{\text{Converges}}$ by ACT $\xleftarrow{\text{ACT}}$

Remember: ACT gets a Conclusion from A.S.
only helpful if A.S. Converges.

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57. $\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{\arctan(2n)}$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\pi}{\arctan(2n)} = 2 \ne 0$$

$\pi/2$

---

58. $\displaystyle\sum_{n=1}^{\infty} 3+\frac{1}{3^n}$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3+\frac{1}{3^n}^{\,0} = 3 \ne 0.$$

59. $\displaystyle\sum_{n=1}^{\infty} e^{1/n}$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\overset{1/n \to 0}{}} = e^0 = 1 \neq 0$$

60. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^6} + \frac{1}{(n+1)^6} = 6\sum_{n=1}^{\infty}\frac{1}{n^6} + \sum_{n=1}^{\infty}\frac{1}{(n+1)^6}$

Constant Multiple of

Convergent p-Series p=6>1

$\underline{\underline{is}}$ Convergent

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)^6} \sim \sum_{n=1}^{\infty}\frac{1}{n^6} \qquad \begin{array}{l}\text{Converges}\\ \text{p-Series}\\ p=6>1\end{array}$$

Bound Terms

$$\frac{1}{(n+1)^6} \leq \frac{1}{n^6} \quad \text{and} \quad \left.\right\rbrace$$

$\Rightarrow$ Series here Converges
by CT

Original Series $\boxed{\text{Converges}}$ b/c
Sum of 2 Convergent Series Converges

61. $\displaystyle\sum_{n=1}^{\infty} \cos^2\left(\frac{\pi n^2 + n}{n^2 + 7}\right)$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos^2\left(\frac{\pi n^2 + n}{n^2 + 7}\;\frac{1/n^2}{1/n^2}\right)$$

$$= \lim_{n\to\infty} \cos^2\left(\frac{\pi + 1/n \to 0}{1 + 7/n \to 0}\right) = \cos^2\pi = (-1)^2 = 1 \neq 0$$

62. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\cos^2(\pi n^2+1)}{n^2+7}$ $\xrightarrow[\text{"Helpful"}]{\text{A.S.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(\pi n^2+1)}{n^2+7} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges p-series $p=2>$

1st

Bound Terms

$$\frac{\cos^2(\pi n^2+1)}{n^2+7} \leq \frac{1}{n^2+7} \leq \frac{1}{n^2} \quad \text{and}$$

work around

$\xrightarrow{\text{ACT}}$

O.S. **Converges** by ACT $\impliedby$ $\implies$ A.S. Converges by CT

[note: O.S. A.C. but not needed here]

Even More Series Note: here must Analyze Absolute Series (AS) b/c Q=A.C., C.C., Diverge?

63. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n+2}$ $\xrightarrow[\text{must}]{\text{A.S.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n+1} \leq \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges Harmonic p-series $p=1$

1st

2nd $\swarrow$ AST

note: CT not helpful here $\frac{1}{5n+1} \leq \frac{1}{n}$

"Smaller than Diverge" not helpful $\hookrightarrow$ inconclusive comparison

① $b_n = \frac{1}{5n+2} > 0$

LCT Limit

② $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{5n+2} = 0$

$\displaystyle\lim_{n\to\infty} \frac{\frac{1}{5n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+1} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{5+\frac{1}{n}} = \frac{1}{5}$

③ Terms decreasing

Finite, Non-zero

$b_{n+1} = \frac{1}{5(n+1)+2} \leq \frac{1}{5n+2} = b_n$

$\underbrace{\qquad}_{5n+7}$

O.S. Converges by AST

$\implies$ A.S. also **Diverges** by LCT

$\hookrightarrow$ Not A.C. $\hookrightarrow$ Test O.S.

$\begin{bmatrix} \text{OR} & f(x) = \frac{1}{5x+2} \\ & \hookrightarrow f'(x) = \frac{-5}{(5x+2)^2} < 0 \\ & \qquad\qquad \oplus \end{bmatrix}$

O.S. **Conditionally Convergent** by Definition

C.C.

64. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3+6n}{n^8+1}$ $\xrightarrow[\text{must}]{\text{A.S.}}$ **1st** $\displaystyle\sum_{n=1}^{\infty} \frac{\boxed{n^3}+6n}{\boxed{n^8}+1} \sim \sum_{n=1}^{\infty} \frac{n^3}{n^8} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ Converges p-Series $p=5>1$

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{n^3+6n}{n^8+1}}{\frac{1}{n^5}} \overset{n^5}{\curvearrowright} = \lim_{n\to\infty} \frac{n^8+6n^6}{n^8+1} \frac{1/n^8}{1/n^8} = \lim_{n\to\infty} \frac{1+\frac{6}{n^2}^{\nearrow 0}}{1+\frac{1}{n^8}_{\searrow 0}} = 1$$

Finite, Non-zero

$\Rightarrow$ $\boxed{\text{A.S. Converges}}$ by LCT

$\llcorner$ O.S. $\boxed{\text{Absolutely Convergent}}$ by Definition

$\boxed{\text{A.C.}}$ Done. Not necessary to analyze O.S.

65. $\displaystyle\sum_{n=1}^{\infty} \frac{5^{2n}}{(2n+1)!\,\ln n}$

R.T.

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty}$ $\oplus$ terms $\left| \dfrac{\dfrac{5^{2(n+1)}}{[2(n+1)+1]!\,\ln(n+1)}}{\dfrac{5^{2n}}{(2n+1)!\,\ln n}} \right|$ $\dfrac{(2n+1)!\,\ln n}{5^{2n}}$

$5^{2n}\cdot 5^2$

$= \lim_{n\to\infty} \dfrac{5^{2n+2}}{5^{2n}} \cdot \dfrac{\ln n}{\ln(n+1)} \cdot \dfrac{(2n+1)!}{(2n+3)!}$

$(*)$

$(2n+3)(2n+2)(2n+1)!$

$= \lim_{n\to\infty} \dfrac{25}{(2n+3)(2n+2)}^{\nearrow 0} = 0 < 1$ $\boxed{\text{A.C.}}$

$\Rightarrow$ O.S. $\boxed{\text{Absolutely Convergent}}$ by R.T.

$(\star)$ $\displaystyle\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}\overset{\infty/\infty}{=}\lim_{x\to\infty}\frac{\ln x}{\ln(x+1)}\underset{L'H}{=}\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{(x+1)}}=\lim_{x\to\infty}\frac{x+1}{x}\overset{\infty/\infty}{=}\underset{L'H}{\lim_{x\to\infty}}\frac{1}{1}=1$

$$\left[\text{or Divide }\frac{1/x}{1/x}\right]$$

66. $\displaystyle\sum_{n=1}^{\infty}\frac{n!\;n^{6}\cdot n^{n}}{10^{4n}\,e^{2n}}$ $\oplus$ both! $\dfrac{10^{4n}\cdot e^{2n}}{n!\cdot n^{6}\cdot n^{n}}$

R.T.

$L=\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\dfrac{\dfrac{(n+1)!\,(n+1)^{6}\,(n+1)^{n+1}}{10^{4(n+1)}\,e^{2(n+1)}}}{\dfrac{n!\,n^{6}\,n^{n}}{10^{4n}\,e^{2n}}}\right|$

$=\displaystyle\lim_{n\to\infty}\;\frac{(n+1)!}{n!}\cdot\frac{(n+1)^{6}}{n^{6}}\cdot\frac{(n+1)^{n+1}}{n^{n}}\cdot\frac{10^{4n}}{10^{4n+4}}\cdot\frac{e^{2n}}{e^{2n+2}}$

$(n+1)\,n!$ $(n+1)^{n}(n+1)$ $\left(\dfrac{n+1}{n}\cdot\dfrac{1/n}{1/n}\right)^{6}$ $10^{4n}\cdot10^{4}$ $e^{2n}\,e^{2}$

extra $(n+1)$ explode

$=\displaystyle\lim_{n\to\infty}\;\frac{(n+1)(n+1)}{10^{4}\cdot e^{2}}\cdot\frac{(n+1)^{n}}{n^{n}}\cdot\left(\frac{1+\frac{1}{n}}{1}\right)^{6}=\infty>1$

$\Rightarrow$ O.S. $\boxed{\text{Diverges}}$ by Ratio Test

67. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{4n+3}$ 1st $\overset{A.S.}{\underset{\text{must}}{\longrightarrow}}$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n+3}\sim\sum_{n=1}^{\infty}\frac{1}{n}$ Diverges, Harmonic p-Series $p=1$

[CT order on Bounds not helpful]

① $b_n = \dfrac{1}{4n+3} > 0$

$\lim\limits_{n \to \infty} \dfrac{\frac{1}{4n+3}}{\frac{1}{n}} = \lim\limits_{n \to \infty} \dfrac{n^{1/n}}{4n+3^{1/n}} = \lim\limits_{n \to \infty} \dfrac{1}{4+\frac{3}{n}} = \dfrac{1}{4}$

Finite, Non-zero

② $\lim\limits_{n \to \infty} b_n = \lim\limits_{n \to \infty} \dfrac{1}{4n+3} = 0$

$\Rightarrow$ A.S. Diverges by LCT

$\hookrightarrow$ Not A.C.
$\hookrightarrow$ Test O.S.

③ Terms decreasing

$b_{n+1} = \dfrac{1}{4(n+1)+3} \le \dfrac{1}{4n+3} = b_n$

$4n+7$

O.S. Converges by AST

OR $f(x) = \dfrac{1}{4x+3}$

$\hookrightarrow f'(x) = \dfrac{-4}{(4x+3)^2} < 0$

$\oplus$

O.S. Conditionally Convergent by Definition

C.C.

Warning: Remember NOT to state C.C. by any Test

68. $\sum\limits_{n=1}^{\infty} \dfrac{(n!)^3 e^{2n}}{(3n)! \, n^n}$

R.T.

$L = \lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n \to \infty} \left| \dfrac{\dfrac{[(n+1)!]^3 \cdot e^{2(n+1)}}{[3(n+1)]! \, (n+1)^{n+1}}}{\dfrac{(n!)^3 \cdot e^{2n}}{(3n)! \, n^n}} \right| \cdot \dfrac{(3n)! \cdot n^n}{(n!)^3 \cdot e^{2n}}$

$$= \lim_{n\to\infty} \frac{[(n+1)!]^3}{[n!]^3} \cdot \frac{e^{2n+2}}{e^{2n}} \cdot \frac{(3n)!}{(3n+3)!} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$\overbrace{(n+1)^3 [n!]^3}$ $\overbrace{e^{2n} \cdot e^2}$

$\underbrace{(3n+3)(3n+2)(3n+1)(3n)!}$ $\overbrace{(n+1)^n (n+1)}$ → $\frac{1}{e}$

$$= \lim_{n\to\infty} \frac{e^{2}}{e} \cdot \left(\frac{n+1}{3n+3}\right)\left(\frac{n+1}{3n+2}\cdot\frac{1/n}{1/n}\right)\left(\frac{n+1}{3n+1}\cdot\frac{1/n}{1/n}\right) \cdot \frac{1}{n+1}$$

$\underbrace{3(n+1)}$

$$= \lim_{n\to\infty} \frac{e}{3} \left(\frac{1+1/n}{3+2/n}\right)\left(\frac{1+1/n}{3+1/n}\right) \cdot \frac{1}{n+1}$$

$\boxed{\frac{1}{3}}$ $\boxed{1/3}$ extra (n+1) makes L→0

$$= \lim_{n\to\infty} \frac{e}{3} \left(\frac{1}{n+1}\right) = 0 < 1$$

$\Rightarrow$ O.S. $\boxed{A.C.}$ by R.T.

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69. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^7+n}$ $\xrightarrow[\text{must}]{A.S.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^7+n} \approx \sum \frac{1}{n^7}$ Converges p-series $p=7>1$

not R.T.

$$\lim_{n\to\infty} \frac{\frac{\arctan n}{n^7+n}}{\frac{1}{n^7}} = \lim_{n\to\infty} \arctan n \left[\frac{n^7}{n^7+n} \cdot \frac{1/n^7}{1/n^7}\right]$$

$$= \lim_{n\to\infty} \arctan n \left[\frac{1}{1+1/n^6}\right] = \frac{\pi}{2}$$

$$\left[\text{OR// run CT with } \frac{\pi}{2} \sum \frac{1}{n^7} \cdot \begin{array}{l}\text{Const Mult} \\ \text{of Conv.} \\ \text{p-series} \\ p=7>1 \\ \text{is Conv.}\end{array}\right.$$

$$\frac{\arctan n}{n^7+n} \le \frac{\pi/2}{n^7+n} < \frac{\pi/2}{n^7} \left.\uparrow\right]$$

$$\Rightarrow \boxed{\text{A.S. Converges}} \text{ by LCT}$$

$$\Rightarrow \text{O.S. } \boxed{\text{A.C.}} \text{ by Definition}$$

70. $$\sum_{n=1}^{\infty} \frac{(-1)^n (3n)! \, n^2}{8^n \cdot (n!)^2 \cdot n^n}$$

R.T.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}\left[3(n+1)!\right](n+1)^2}{8^{n+1}\left[(n+1)!\right]^2 (n+1)^{n+1}}}{\dfrac{(-1)^n (3n)! \, n^2}{8^n (n!)^2 n^n}} \right|$$

$$\overbrace{(3n+3)(3n+2)(3n+1)}(3n)!$$

$$= \lim_{n\to\infty} \frac{(3n+3)!}{(3n)!} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{8^n}{8^{n+1}} \cdot \frac{[n!]^2}{[(n+1)!]^2} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$\left(\frac{n+1}{n}\right)^2 \qquad 8^n \cdot 8 \qquad \frac{(n+1)^2[n!]^2}{(n+1)^n (n+1)}$$

$$3(n+1)$$

$$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)^{1/n}}{(n+1)\,(n+1)\,(n+1)^{1/n}} \cdot \frac{1}{8} \cdot \frac{n^n}{(n+1)^n} \quad \frac{1}{e}$$

$$= \lim_{n\to\infty} \frac{3}{8} \left[\frac{3+\cancel{3}/n}{1+1/n}\right]^{\to 3} \cdot \left[\frac{3+1/n}{1+1/n}\right]^{\to 3} \cdot \frac{1}{e} = \frac{27}{8e} > 1$$

$$\Rightarrow \text{O.S. } \boxed{\text{Diverges}} \text{ by Ratio Test}$$

71. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3+7}{n^7+3}$ $\xrightarrow[\text{must}]{\text{A.S.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+7}{n^7+3}$ $\underset{\sim}{\sim}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^4}$

Converges p-Series
p = 4 > 1

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{n^3+7}{n^7+3}}{\frac{1}{n^4}} \overset{n^4}{=} \lim_{n\to\infty} \frac{n^7+7n^4}{n^7+3} \frac{1/n^7}{1/n^7} = \lim_{n\to\infty} \frac{1+\frac{7}{n^3}\to 0}{1+\frac{3}{n^7}\to 0} = 1$$

$\Rightarrow$ A.S. also Converges by LCT

$\Rightarrow$ O.S. $\boxed{\text{Absolutely Convergent}}$ by Definition

72. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n \cdot \pi^n (2n)!}{n^n \; 4^n \; n!}$

R.T.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1} \ln(n+1) \; \pi^{n+1} [2(n+1)]!}{(n+1)^{n+1} \cdot 4^{n+1} \cdot (n+1)!}}{\dfrac{(-1)^n \cdot \ln n \; \pi^n (2n)!}{n^n \cdot 4^n \cdot n!}} \right|$$

$$= \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \frac{\pi^{n+1}}{\pi^n} \cdot \frac{(2n+2)!}{(2n)!} \cdot \frac{n^n}{(n+1)^{n+1}} \cdot \frac{4^n}{4^{n+1}} \cdot \frac{n!}{(n+1)!}$$

(*)

$\pi^n \cdot \pi$; $(2n+2)(2n+1)(2n)!$

$(n+1)^n (n+1)$; $4^n \cdot 4$; $(n+1) \cdot n!$

$$= \lim_{n \to \infty} \pi \cdot \frac{(2n+2)(2n+1)\frac{1}{n}}{(n+1)(n+1)\frac{1}{n}} \cdot \frac{n^n}{(n+1)^n} \nearrow \frac{1}{e}$$

(with annotation: $2(n+1)$ over $(2n+2)$)

$$= \lim_{n \to \infty} \frac{\pi}{4e} \cdot 2 \left[\frac{2 + \frac{1}{n} \searrow 0}{1 + \frac{1}{n} \searrow 0} \right]^2 = \frac{4\pi}{4e} = \frac{\pi}{e} > 1$$

($\pi \approx 3.14$, $e \approx 2.718\ldots$)

$$\Rightarrow \text{O.S.} \boxed{\text{Diverges}} \text{ by Ratio Test}$$

or R.T.

(★)

$$\lim_{n \to \infty} \frac{\ln(n+1)}{\ln n} \overset{\infty/\infty}{=} \lim_{x \to \infty} \frac{\ln(x+1)}{\ln x} \overset{\infty/\infty}{\underset{\text{L'H}}{=}} \lim_{x \to \infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{x}{x+1} \overset{\infty/\infty}{\underset{\text{L'H}}{=}} \lim_{x \to \infty} \frac{1}{1} = 1$$

or use "Divides by" Trick