Math 121 Exam 2 Fall 2024 Answer Key

1(a) $\displaystyle\int_0^e x^3 \ln x\,dx = \lim_{t\to 0^+}\int_t^e x^3\ln x\,dx = \lim_{t\to 0^+} \frac{x^4}{4}\cdot \ln x \Big|_t^e - \frac{1}{4}\int_t^e x^3\,dx$

IBP

$\boxed{\begin{array}{ll} u = \ln x & dv = x^3\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^4}{4} \end{array}}$

$$= \lim_{t\to 0^+} \frac{x^4}{4}\cdot \ln x \Big|_t^e - \frac{x^4}{16}\Big|_t^e$$

$$= \lim_{t\to 0^+} \frac{e^4}{4}\cdot \overset{1}{\cancel{\ln e}} - \overset{0^+\cdot(-\infty)\ \to\ 0}{\cancel{\frac{t^4}{4}\cdot \ln t}} - \left(\frac{e^4}{16} - \overset{0}{\cancel{\frac{t^4}{16}}}\right)$$

see ☆

$$= \frac{e^4}{4} - \frac{e^4}{16} = \frac{4e^4 - e^4}{16} = \frac{3e^4}{16} \quad \text{Converges}$$

☆ $\displaystyle \lim_{t\to 0^+} \overset{0\cdot(-\infty)}{t^4\cdot \ln t} = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t^4}} \overset{-\infty/\infty}{\underset{\text{L'H}}{=}} \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-4}{t^5}} \overset{-t^5/4}{=} \lim_{t\to 0^+} \overset{0}{\cancel{\frac{-t^4}{4}}} = 0$

Flip

$t^{-4} \to -4t^{-5}$

Key Note: $\ln 0$ is undefined, so must "sneak attack" 0 using Limit $\to 0^+$

1(b) $\displaystyle\int_{-\infty}^{-9} \frac{7}{x^2+4x+53}\,dx = \lim_{t\to-\infty}\int_t^{-9}\frac{7}{x^2+4x+53}\,dx = \lim_{t\to-\infty}\int_t^{-9}\frac{7}{(x+2)^2+49}\,dx$

Complete the Square

Scratch
$(x+2)^2 = x^2+4x+4 \underbrace{}_{+49}$

$$= \lim_{t\to-\infty}\int_{t+2}^{-7}\frac{7}{u^2+49}\,du = \lim_{t\to-\infty} \cancel{7}\left(\frac{1}{\cancel{7}}\right)\arctan\left(\frac{u}{7}\right)\Big|_{t+2}^{-7}$$

$\boxed{\begin{array}{l} u = x+2 \\ du = dx \end{array}}$

$\boxed{\begin{array}{l} x = t \Rightarrow u = t+2 \\ x = -9 \Rightarrow u = -7 \end{array}}$

$$= \lim_{t\to-\infty} \overset{-\frac{\pi}{4}}{\cancel{\arctan\left(\overset{-1}{\cancel{\frac{-7}{7}}}\right)}} - \overset{-\frac{\pi}{2}}{\cancel{\arctan\left(\overset{-\infty}{\cancel{\frac{t+2}{7}}}\right)}}$$

$$= -\frac{\pi}{4} + \underset{\frac{2\pi}{4}}{\frac{\pi}{2}} = \frac{\pi}{4} \quad \text{Converges}$$

1(c) $\displaystyle\int_{-5}^{-4} \frac{7-x}{x^2+4x-5}\,dx = \int_{-5}^{-4} \frac{7-x}{(x-1)(x+5)}\,dx = \lim_{t\to -5^+} \int_t^{-4} \frac{7-x}{(x-1)(x+5)}\,dx$

PFD

$\displaystyle (x-1)(x+5)\left(\frac{7-x}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5}\right)(x-1)(x+5)$

$7-x = A(x+5)+B(x-1)$
$= Ax+5A+Bx-B$
$= (A+B)x+(5A-B)$

Conditions
- $A+B=-1 \Rightarrow A=-1-B$
- $5A-B=7$ $\quad 5(-1-B)-B=7$
  $-5-5B-B=7$
  $-6B=12$
  $B=-2$

$A=-1-(-2)$
$=-1+2=1$

$\displaystyle = \lim_{t\to -5^+} \int_t^{-4} \frac{1}{x-1} - \frac{2}{x+5}\,dx$

$\displaystyle = \lim_{t\to -5^+} \ln|x-1| - 2\ln|x+5| \Big|_t^{-4}$

$\displaystyle = \lim_{t\to -5^+} \underbrace{\ln|-5|}_{\text{finite}}^{\ln 5} - \underbrace{2\ln|1|}_{\text{finite}}^{0} - \left(\underbrace{\ln|t-1|}_{\text{finite}}^{\ln 6} - 2\ln|t+5|\right)$ $\quad t \to -5^+,\ t+5 \to 0^+,\ -\infty$

$= -\infty$ Diverges

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^7}$ $\longrightarrow$ study Related Function $f(x) = \dfrac{\ln x}{x^7}$

$\displaystyle\int_1^{\infty} \frac{\ln x}{x^7}\,dx = \lim_{t\to\infty} \int_1^t (\ln x)\cdot x^{-7}\,dx \overset{\text{IBP}}{=} \lim_{t\to\infty} -\frac{\ln x}{6x^6}\Big|_1^t + \frac{1}{6}\int_1^t x^{-7}\,dx$ $\quad \dfrac{x^{-6}}{-6}$

$$\boxed{\begin{array}{ll} u=\ln x & dv = x^{-7}dx \\ du = \frac{1}{x}dx & v = \frac{x^{-6}}{-6} = -\frac{1}{6x^6} \end{array}}$$

$\displaystyle = \lim_{t\to\infty} -\frac{\ln x}{6x^6}\Big|_1^t - \frac{1}{36x^6}\Big|_1^t$

$\displaystyle = \lim_{t\to\infty} \underbrace{\frac{-\ln t}{6t^6}}_{\text{L'H here}} + \frac{\ln 1}{6} - \left(\frac{1}{36t^6} - \frac{1}{36}\right)$ $\quad \frac{-\infty}{\infty},\ 0,\ 0$

$\displaystyle \overset{\text{L'H}}{=} \lim_{t\to\infty} \frac{-\frac{1}{t}}{36t^5} + \frac{1}{36}$ $\quad \dfrac{1}{36t^5}$

$\displaystyle = \lim_{t\to\infty} \frac{-1}{36t^6} + \frac{1}{36} = \frac{1}{36}$ Integral Converges

$\Rightarrow$ $\boxed{\text{Series Converges by Integral Test}}$

3(a) $\displaystyle\sum_{n=2}^{\infty} \underbrace{\frac{e^{3n}}{7\ln n}}_{a_n}$ → Ratio Test OR nTDT

Option 1. Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{e^{3(n+1)}}{7\ln(n+1)}}{\dfrac{e^{3n}}{7\ln n}}\right| = \lim_{n\to\infty}\left(\frac{e^{3n+3}}{\cancel{e^{3n}}}\right)\left(\frac{\cancel{7}\ln n}{\cancel{7}\ln(n+1)}\right) = e^3 > 1$$

$\frac{7\ln n}{e^{3n}}$ + term $\cancel{e^{3n}} \cdot e^3$ $\frac{\infty}{\infty}$ $\to 1$

⟹ $\boxed{\text{Series Diverges by Ratio Test}}$

☆ $$\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty}\frac{\ln x}{\ln(x+1)} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{n\to\infty}\frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty}\frac{x+1}{x} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{1}{1} = 1$$

OR $= \lim_{x\to\infty} 1 + \cancel{\frac{1}{x}}^{\,0} = 1$

OR

Option 2: $\boxed{\text{Diverges by nTDT}}$

$$\lim_{n\to\infty}\frac{e^{3n}}{7\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{e^{3x}}{7\ln x} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{3e^{3x}}{\frac{7}{x}} = \lim_{x\to\infty}\frac{3xe^{3x}}{7} = \infty \neq 0$$

$\frac{x}{7}$

3(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(3n+7)!}$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{1}{(3(n+1)+7)!}}{\dfrac{1}{(3n+7)!}}\right| = \lim_{n\to\infty}\frac{\cancel{(3n+7)!}}{(3n+10)!} = \lim_{n\to\infty}\frac{1}{(3n+10)(3n+9)(3n+8)} = 0 < 1$$

$(3n+10)! = (3n+10)(3n+9)(3n+8)\cancel{(3n+7)!}$

⟹ Absolutely Convergent by the Ratio Test

⟹ Since the Original Series is already the Absolute Series, then

Absolute Convergence = $\boxed{\text{Convergence}}$

OR Converges by ACT since Absolute Convergence implies Convergence

Factor out

3(c) $-\frac{3}{\sqrt{1}}-\frac{3}{\sqrt{2}}-\frac{3}{\sqrt{3}}-\frac{3}{\sqrt{4}}-\frac{3}{\sqrt{5}}-\frac{3}{\sqrt{6}}\cdots = -3\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$ Note: not Alternating here

Constant Multiple of the Divergent p-Series with $p=\frac{1}{2}<1$ is $\boxed{\text{Divergent}}$

3(d) $$\sum_{n=1}^{\infty}\frac{\ln 3}{n^7}+\frac{(-3)^n}{7^{2n+1}} = \sum_{n=1}^{\infty}\frac{\ln 3}{n^7} + \sum_{n=1}^{\infty}\frac{(-3)^n}{7^{2n+1}}$$

$$\ln 3\sum_{n=1}^{\infty}\frac{1}{n^7} \quad + \quad -\frac{3}{7^3}+\frac{3^2}{7^5}-\frac{3^3}{7^7}+\cdots \qquad r=\frac{-3}{7^2}$$

Constant Multiple of Convergent p-Series $p=7>1$ is Convergent

Converges by Geometric Series Test with $|r|=\left|-\frac{3}{7^2}\right|=\frac{3}{49}<1$

Original Series Converges because the Sum of Two Convergent Series is Convergent

3(e) $\sum_{n=1}^{\infty}\frac{(-1)^n 3}{n^7} \longrightarrow$ AST OR ACT

Option 1: AST

1. choose $b_n=\frac{3}{n^7}>0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{3}{n^7}=0$
3. Terms Decreasing

$b_{n+1}=\frac{3}{(n+1)^7}\le\frac{3}{n^7}=b_n$

Series $\boxed{\text{Converges}}$ by the Alternating Series Test

OR

Option 2: ACT

O.S. $\sum_{n=1}^{\infty}\frac{(-1)^n 3}{n^7} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty}\frac{3}{n^7} = 3\sum_{n=1}^{\infty}\frac{1}{n^7}$

Absolute Series Converges because a Constant Multiple of a Convergent p-Series $p=7>1$ is Convergent

ACT

Original Series Converges by the Absolute Convergence Test

3(f) $\displaystyle\sum_{n=2}^{\infty}\left(1-\frac{7}{n^3}\right)^{n^3}$ Diverges by nTDT because

$$\lim_{n\to\infty}\left(1-\frac{7}{n^3}\right)^{n^3} \overset{1^\infty}{=} \lim_{x\to\infty}\left(1-\frac{7}{x^3}\right)^{x^3} = e^{\lim_{x\to\infty}\ln\left(\left(1-\frac{7}{x^3}\right)^{x^3}\right)} = e^{\lim_{x\to\infty} x^3 \ln\left(1-\frac{7}{x^3}\right)}$$

$$= e^{\lim_{x\to\infty}\frac{\ln\left(1-\frac{7}{x^3}\right)}{\frac{1}{x^3}}} \underset{L'H}{\overset{0/0}{=}} e^{\lim_{x\to\infty}\frac{\frac{1}{1-\cancel{\frac{7}{x^3}}}\cdot\left(\cancel{\frac{21}{x^4}}\right)}{\cancel{-\frac{3}{x^4}}}} = e^{-7} \neq 0$$

$x^{-3}\to -3x^{-4}$

$-7x^{-3}\to +21x^{-4}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(3n+7)^7} \xrightarrow{A.S.} \sum_{n=1}^{\infty}\frac{1}{(3n+7)^7} \approx \sum_{n=1}^{\infty}\frac{1}{n^7}$ Converges p-Series $p=7>1$

Bound Terms

$$\frac{1}{(3n+7)^7} \le \frac{1}{n^7}$$

$\Rightarrow$ Absolute Series also Converges by the Comparison Test

$\overset{ACT}{\Leftarrow}$ Original Series Converges by the Absolute Convergence Test

Absolutely Convergent example

5(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^3+7}{n^7+3} \longrightarrow \sum_{n=1}^{\infty}\frac{n^3+7}{n^7+3} \approx \sum_{n=1}^{\infty}\frac{n^3}{n^7} = \sum_{n=1}^{\infty}\frac{1}{n^4}$ Converges p-Series $p=4>1$

LCT Limit

$$\lim_{n\to\infty}\frac{\frac{n^3+7}{n^7+3}}{\frac{1}{n^4}} = \lim_{n\to\infty}\frac{n^7+7n^4}{n^7+3}\cdot\frac{\frac{1}{n^7}}{\frac{1}{n^7}} = \lim_{n\to\infty}\frac{1+\cancel{\frac{7}{n^3}}}{1+\cancel{\frac{3}{n^7}}} = 1 \quad \text{Finite Non-Zero}$$

$\Rightarrow$ Absolute Series also Converges by Limit Comparison Test

$\Rightarrow$ Original Series is Absolutely Convergent (by Definition)

$$5(b)\quad \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1)!\, 3^n}{n^7 (n!)\, n^n} \qquad \text{Ratio Test}$$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}} (2(n+1))!\, 3^{n+1}}{(n+1)^7 (n+1)!\,(n+1)^{n+1}}}{\dfrac{\cancel{(-1)^n} (2n)!\, 3^n}{n^7 (n!)\cdot n^n}} \right|$$

$$\left(\frac{n}{n+1}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)^7 \quad \left(\frac{1}{1+\frac{1}{n}}\right)^7 \to 1$$

$$(2n+2)(2n+1)\cancel{(2n)!} \qquad \cancel{3^n}\cdot 3$$

see above $\to 1$

$$= \lim_{n\to\infty} \left(\frac{(2n+2)!}{\cancel{(2n)!}}\right)\left(\frac{3^{n+1}}{\cancel{3^n}}\right)\left(\frac{n^7}{(n+1)^7}\right)\left(\frac{\cancel{n!}}{(n+1)!}\right)\left(\frac{n^n}{(n+1)^{n+1}}\right)$$

$$(n+1)\cancel{n!} \qquad (n+1)^n\cdot(n+1)$$

$$2\cancel{(n+1)}$$

$$= \lim_{n\to\infty} 3\left(\frac{2n+2}{\cancel{n+1}}\right)\left(\frac{2n+1}{n+1}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{n^n}{(n+1)^n}\right) = \frac{12}{e} > 1$$

$$\frac{2+\frac{1}{n}}{1+\frac{1}{n}} \to 2 \qquad \frac{n^n}{(n+1)^n} \to \frac{1}{e}$$

$\Rightarrow$ $\boxed{\text{Series Diverges by Ratio Test}}$

$$5(c)\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{3n+7} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty} \frac{1}{3n+7} \approx \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{Diverges p-Series } p=1$$

AST

$$\lim_{n\to\infty} \frac{\frac{1}{3n+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n+7}\,\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{3+\frac{7}{n}} = \frac{1}{3} \quad \text{Finite Non-Zero}$$

1. Pick $b_n = \frac{1}{3n+7} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3n+7} = 0$
3. Terms Decreasing

$$b_{n+1} = \frac{1}{\underbrace{3(n+1)+7}_{3n+10}} \le \frac{1}{3n+7} = b_n$$

$\Rightarrow$ Absolute Series also Diverges by the Limit Comparison Test

Original Series Converges by the Alternating Series Test

Finally, the Original Series is Conditionally Convergent by Definition

OR show Related Function $f(x) = \frac{1}{3x+7}$ has $f'(x) = \frac{-3}{(3x+7)^2} < 0$

$\oplus$

Bonus: First, show that the Series $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln(\ln n))\cdot 5^n (n!)^3 (2n)!}{n^{2n}\cdot (3n)!}$ Converges by the Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(\ln(\ln(n+1)))\cdot 5^{n+1}((n+1)!)^3(2(n+1))!}{(n+1)^{2(n+1)}\;(3(n+1))!}}{\dfrac{(\ln(\ln n))\cdot 5^n (n!)^3 (2n)!}{n^{2n}\;(3n)!}}\right|$$

Flip + Multiply

See below ☆

$$= \lim_{n\to\infty} \underbrace{\left(\frac{\ln(\ln(n+1))}{\ln(\ln n)}\right)}_{1}\left(\frac{5^{n+1}}{\cancel{5^n}}\right)\left(\frac{((n+1)!)^3}{(n!)^3}\right)\left(\frac{(2n+2)!}{(2n)!}\right)\left(\frac{n^{2n}}{(n+1)^{2n+2}}\right)\left(\frac{(3n)!}{(3n+3)!}\right)$$

$5^{n+1} = \cancel{5^n}\cdot 5 \qquad ((n+1)!)^3 = (n+1)^3\cancel{(n!)^3} \qquad (2n+2)! = (2n+2)(2n+1)\cancel{(2n)!}$

$(n+1)^{2n+2} = (n+1)^{2n}\cdot(n+1)^2 \qquad (3n+3)! = (3n+3)(3n+2)(3n+1)\cancel{(3n)!}$

$$= \lim_{n\to\infty} \frac{5\,(n+1)^3\,(2n+2)(2n+1)}{(n+1)^2\,(3n+3)(3n+2)(3n+1)}\Bigg(\underbrace{\frac{n^n}{(n+1)^n}}_{\frac{1}{e}}\Bigg)^{2} \qquad \frac{1}{e^2}$$

$(n+1)^3/(n+1)^2 = n+1 \qquad 2n+2 = 2(n+1) \qquad 3n+3 = 3(n+1)$

$$= \lim_{n\to\infty} \frac{10}{3e^2}\left(\frac{n+1}{3n+2}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right)\left(\frac{2n+1}{3n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$$= \lim_{n\to\infty} \frac{10}{3e^2}\underbrace{\left(\frac{1+\frac{1}{n}}{3+\frac{2}{n}}\right)}_{\frac{1}{3}}\underbrace{\left(\frac{2+\frac{1}{n}}{3+\frac{1}{n}}\right)}_{\frac{2}{3}} = \frac{20}{27e^2} < 1$$

$\Rightarrow$ The Series is (Absolutely) Convergent by the Ratio Test

☆ $$\lim_{x\to\infty}\frac{\ln(\ln(x+1))}{\ln(\ln x)} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{\frac{1}{\ln(x+1)}\cdot\frac{1}{(x+1)}}{\frac{1}{\ln x}\cdot\frac{1}{x}} = \lim_{x\to\infty}\underbrace{\frac{\ln x}{\ln(x+1)}}_{1\ ☆}\cdot\underbrace{\frac{x}{x+1}}_{1\ ☆} = 1$$

See below

☆ $$\lim_{x\to\infty}\frac{\ln x}{\ln(x+1)} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty}\frac{x+1}{x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{1}{1} = 1$$

☆ $$\lim_{x\to\infty}\frac{x}{x+1} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{1}{1} = 1$$

Bonus (continued)

$\Rightarrow$ the Sequence terms must Approach 0 as $n \to \infty$ because otherwise if $\lim\limits_{n\to\infty} \overset{\text{terms}}{a_n} \neq 0$, then the Series would Diverge by nTDT which would contradict what we proved above... the Series Converges.

That is, since $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(\ln n)\, 5^n (n!)^3 (2n)!}{n^{2n}\,(3n)!}$ Converges then

$$\lim_{n\to\infty} \frac{(\ln(\ln n))\, 5^n \cdot (n!)^3 (2n)!}{n^{2n}\,(3n)!} = 0 \Rightarrow \text{Sequence Converges.}$$