Exam 1 Fall 2024 Answer Key

1(a)

$$\lim_{x\to 0}\frac{\cos(3x)-\arctan(2x)+2x-1}{e^{-4x}-1+4x}\;\stackrel{L'H}{=}\;\lim_{x\to 0}\frac{-3\sin(3x)-\dfrac{2}{1+(2x)^2}+2}{-4e^{-4x}+4}\qquad \tfrac{0}{0}$$

$$\stackrel{prep}{=}\;\lim_{x\to 0}\frac{-3\sin(3x)-2(1+4x^2)^{-1}+2}{-4e^{-4x}+4}\qquad \tfrac{0}{0}$$

$$\stackrel{L'H}{=}\;\lim_{x\to 0}\frac{-9\cos(3x)+2(1+4x^2)^{-2}\cdot(8x)}{16e^{-4x}}$$

$$\stackrel{rewrite}{=}\;\lim_{x\to 0}\frac{-9\cos(3x)+\dfrac{16x}{(1+4x^2)^2}}{16e^{-4x}}=\boxed{-\frac{9}{16}}$$

1(b)

$$\lim_{x\to 0^+}\sqrt{x}\,\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}}\;\stackrel{L'H}{=}\;\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}}=\lim_{x\to 0^+}-2\sqrt{x}=\boxed{0}$$

$0\cdot(-\infty)$, $\frac{-\infty}{\infty}$

$\frac{d}{dx}x^{-1/2}=-\frac{1}{2}x^{-3/2}$

1(c)

$$\lim_{x\to\infty}\left(1-\arcsin\left(\frac{2}{x^6}\right)\right)^{x^6}=e^{\lim_{x\to\infty}\ln\left(\left(1-\arcsin\left(\frac{2}{x^6}\right)\right)^{x^6}\right)}$$

$$=e^{\lim_{x\to\infty}x^6\cdot\ln\left(1-\arcsin\left(\frac{2}{x^6}\right)\right)}\qquad \infty\cdot 0$$

$$=e^{\lim_{x\to\infty}\frac{\ln\left(1-\arcsin\left(\frac{2}{x^6}\right)\right)}{\frac{1}{x^6}}}\qquad \tfrac{0}{0}$$

$2x^{-6}$, $-6x^{-7}$

$$=e^{\lim_{x\to\infty}\frac{\frac{1}{1-\arcsin\left(\frac{2}{x^6}\right)}\cdot\frac{-1}{\sqrt{1-\left(\frac{2}{x^6}\right)^2}}\cdot\left(\frac{-12}{x^7}\right)}{\frac{-6}{x^7}}}$$

$$=e^{1\cdot(-1)\cdot 2}=\boxed{e^{-2}}$$

2. $$\int_{-2}^{2}\sqrt{4-x^2}\,dx = \int_{x=-2}^{x=2}\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta = 4\int_{x=-2}^{x=2}\cos^2\theta\,d\theta$$

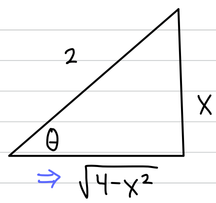
$$\sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)} = \cancel{\sqrt{4\cos^2\theta}} = 2\cos\theta$$

Trig. Sub $\boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta \end{aligned}}$

$\sin\theta = \frac{x}{2}$

$\theta = \arcsin\left(\frac{x}{2}\right)$

$$= 4\int_{x=-2}^{x=2}\frac{1+\cos(2\theta)}{2}\,d\theta = \frac{4}{2}\int_{x=-2}^{x=2}1+\cos(2\theta)\,d\theta$$



$$= 2\left(\theta + \frac{\overbrace{\sin(2\theta)}^{\cancel{2}\sin\theta\cos\theta}}{\cancel{2}}\right)\Bigg|_{x=-2}^{x=2} = 2\left(\arcsin\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)\frac{\sqrt{4-x^2}}{2}\right)\Bigg|_{-2}^{2}$$

$$= 2\left(\underbrace{\arcsin\left(\tfrac{2}{2}\right)}_{\frac{\pi}{2}} + \left(\frac{2}{2}\right)\underbrace{\frac{\sqrt{4-4}}{2}}_{0} - \left(\underbrace{\arcsin\left(\tfrac{-2}{2}\right)}_{-\frac{\pi}{2}} + \left(\frac{-2}{2}\right)\underbrace{\frac{\sqrt{4-4}}{2}}_{0}\right)\right)$$

$$= 2\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 2\pi$$

3. $$\int_{0}^{\ln\sqrt{3}}\frac{e^x}{\sqrt{4-e^{2x}}}\,dx = \int_{0}^{\ln\sqrt{3}}\frac{e^x}{\sqrt{4-(e^x)^2}}\,dx = \int_{1}^{\sqrt{3}}\frac{1}{\sqrt{4-u^2}}\,du = \arcsin\left(\frac{u}{2}\right)\Bigg|_{1}^{\sqrt{3}}$$

$\boxed{\begin{aligned} u &= e^x \\ du &= e^x\,dx \end{aligned}}$ $\boxed{\begin{aligned} x=0 &\Rightarrow u=e^0=1 \\ x=\ln\sqrt{3} &\Rightarrow u=e^{\ln\sqrt{3}}=\sqrt{3} \end{aligned}}$

$$= \underbrace{\arcsin\left(\frac{\sqrt{3}}{2}\right)}_{\frac{\pi}{3}} - \underbrace{\arcsin\left(\frac{1}{2}\right)}_{\frac{\pi}{6}}$$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

4. $$\int_{e}^{e^3}\frac{1}{x\left(3+(\ln x)^2\right)}\,dx = \int_{1}^{3}\frac{1}{3+u^2}\,du = \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_{1}^{3}$$

$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$ $\boxed{\begin{aligned} x=e &\Rightarrow u=\ln e=1 \\ x=e^3 &\Rightarrow u=\ln e^3=3 \end{aligned}}$

$$= \frac{1}{\sqrt{3}}\left(\underbrace{\arctan\left(\frac{3}{\sqrt{3}}\right)}_{\frac{\pi}{3}} - \underbrace{\arctan\left(\frac{1}{\sqrt{3}}\right)}_{\frac{\pi}{6}}\right), \quad \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{6\sqrt{3}}$$

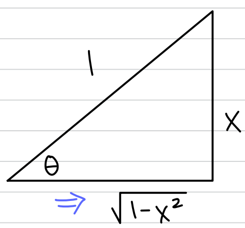
5. $$\int x\arcsin x\,dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int\frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot\cancel{\cos\theta}\,d\theta$$

$$\sqrt{\cos^2\theta} \rightarrow \cancel{\cos\theta}$$

$$\boxed{\begin{array}{ll} u=\arcsin x & dv = x\,dx \\ du=\dfrac{1}{\sqrt{1-x^2}}\,dx & v=\dfrac{x^2}{2}\end{array}}$$

$$\boxed{\begin{array}{l} x=\sin\theta \\ dx=\cos\theta\,d\theta\end{array}}$$

$\theta = \arcsin x$



$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int\sin^2\theta\,d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int\frac{1-\cos(2\theta)}{2}\,dx$$

$\cancel{2}\sin\theta\cos\theta$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\theta - \frac{\sin(2\theta)}{\cancel{2}}\right) + C$$

$$\boxed{= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\arcsin x - x\sqrt{1-x^2}\right) + C}$$

OR// $$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C$$

6. $$\int\frac{1}{(9+x^2)^{7/2}}\,dx = \int\frac{1}{\left(\sqrt{9+x^2}\right)^7}\,dx = \int\frac{1}{\left(\sqrt{9+9\tan^2\theta}\right)^7}\cdot 3\sec^2\theta\,d\theta$$

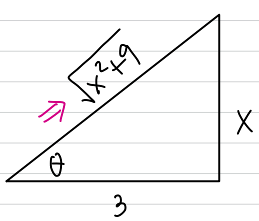
$$\left(\sqrt{9(1+\tan^2\theta)}\right)^7$$

$$\left(\cancel{\sqrt{9\sec^2\theta}}\right)^7$$

$$(3\sec\theta)^7$$

$$\boxed{\begin{array}{l} x = 3\tan\theta \\ dx = 3\sec^2\theta\,d\theta\end{array}}$$

$\tan\theta = \dfrac{x}{3}$



$$= \int\frac{1}{3^7\cdot\sec^7\theta}\cdot 3\sec^2\theta\,d\theta = \frac{1}{3^6}\int\frac{1}{\sec^5\theta}\,d\theta$$

$(\cos^2\theta)^2$

$$= \frac{1}{729}\int\cos^5\theta\,d\theta = \frac{1}{729}\int\cos^4\theta\cdot\cos\theta\,d\theta = \frac{1}{729}\int(1-\sin^2\theta)^2\cdot\cos\theta\,d\theta$$

Given

$$= \frac{1}{729}\int(1-u^2)^2\,du = \frac{1}{729}\int 1-2u^2+u^4\,du = \frac{1}{729}\left(u-\frac{2}{3}u^3+\frac{u^5}{5}\right)+C$$

$$\boxed{\begin{array}{l} u=\sin\theta \\ du=\cos\theta\,d\theta\end{array}}$$

$$= \frac{1}{729}\left(\sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta\right)+C$$

$$\boxed{= \frac{1}{729}\left(\frac{x}{\sqrt{x^2+9}} - \frac{2}{3}\left(\frac{x}{\sqrt{x^2+9}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{x^2+9}}\right)^5\right)+C}$$

7. $\displaystyle\int \ln(x^2+9)\,dx = \int \ln(x^2+9)\cdot 1\,dx = x\ln(x^2+9) - 2\int \frac{x^2+9-9}{x^2+9}\,dx$ slip-in / slip-out

$u = \ln(x^2+9)$	$dv = 1\,dx$
$du = \frac{1}{x^2+9}\cdot(2x)\,dx$	$v = x$

$$= x\ln(x^2+9) - 2\int \frac{x^2+9-9}{x^2+9}\,dx$$ split-split

$$= x\ln(x^2+9) - 2\left(\int \frac{\cancel{x^2+9}}{\cancel{x^2+9}} - \frac{9}{x^2+9}\,dx\right)$$

a-rule

$$\boxed{= x\ln(x^2+9) - 2\left(x - 9\cdot\left(\frac{1}{3}\right)\arctan\left(\frac{x}{3}\right)\right) + C}$$

OR

$$\boxed{= x\ln(x^2+9) - 2x + 3\arctan\left(\frac{x}{3}\right) + C}$$