Math 121 Final Exam Spring 2022

1(a) $$\lim_{x\to0}\frac{5x^2+\arctan(3x)-3x}{2x-\sin(2x)-3x^2}=\lim_{x\to0}\frac{5x^2+\sum_{n=0}^{\infty}\frac{(-1)^n(3x)^{2n+1}}{2n+1}-3x}{2x-\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{(2n+1)!}-3x^2}$$

$$=\lim_{x\to0}\frac{5x^2+\cancel{3x}-\frac{(3x)^3}{3}+\frac{(3x)^5}{5}-\frac{(3x)^7}{7}+\dots\cancel{-3x}}{\cancel{2x}-\left(\cancel{2x}-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\frac{(2x)^7}{7!}+\dots\right)-3x^2}$$

$$=\lim_{x\to0}\frac{5x^2-\frac{\cancel{27}^{\,9}x^3}{\cancel{3}}+\frac{3^5x^5}{5}-\frac{3^7x^7}{7}+\dots}{-3x^2+\frac{8x^3}{3!}-\frac{32x^5}{5!}+\dots}\qquad \frac{0}{0}\qquad \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$=\lim_{x\to0}\frac{5-\cancel{9x+\frac{3^5x^3}{5}-\frac{3^7x^5}{7}+\dots}\to0}{-3+\cancel{\frac{8x}{3!}-\frac{32x^3}{5!}+\dots}\to0}=\boxed{-\frac{5}{3}}$$

1(b) $$\lim_{x\to0}\frac{5x^2+\arctan(3x)-3x}{2x-\sin(2x)-3x^2}\ \underset{L'H}{\overset{0/0}{=}}\ \lim_{x\to0}\frac{10x+\frac{3}{1+(3x)^2}-3}{2-2\cos(2x)-6x}\quad \frac{0}{0}$$

$3(1+9x^2)^{-1}$

↳ $-3(1+9x^2)^{-2}(18x)$

$$\underset{L'H}{=}\lim_{x\to0}\frac{10-\cancel{\frac{54x}{(1+9x^2)^2}}\to0}{4\cancel{\sin(2x)}-6}=\frac{10}{-6}=\boxed{-\frac{5}{3}}\quad \text{Match!}$$

2(a) $$\int\frac{1}{(x^2+4)^{7/2}}\,dx=\int\frac{1}{\left(\sqrt{x^2+4}\right)^7}\,dx=\int\frac{1}{\left(\sqrt{4\tan^2\theta+4}\right)^7}\cdot2\sec^2\theta\,d\theta$$

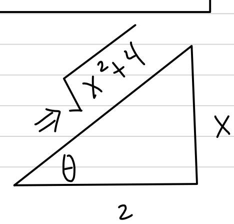
$4(\tan^2\theta+1)$

$\left(\sqrt{4\sec^2\theta}\right)^7$

Trig. Sub

$$\boxed{\begin{aligned}x&=2\tan\theta\\dx&=2\sec^2\theta\,d\theta\end{aligned}}$$

↳ $\tan\theta=\frac{x}{2}$



$$=\int\frac{1}{(2\sec\theta)^7}\cdot2\sec^2\theta\,d\theta=\frac{\cancel{2}}{2^{\cancel{7}}_{\,6}}\int\frac{\cancel{\sec^2\theta}}{\cancel{\sec^7\theta}}_{\,\sec^5\theta}\,d\theta$$

$$= \frac{1}{64}\int \frac{1}{\sec^5\theta}\,d\theta = \frac{1}{64}\int \cos^5\theta\,d\theta$$

$$= \frac{1}{64}\int \overbrace{\cos^4\theta}^{(\cos^2\theta)^2}\cdot\cos\theta\,d\theta = \frac{1}{64}\int (1-\sin^2\theta)^2\cos\theta\,d\theta$$

$$\boxed{\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}}$$

$$= \frac{1}{64}\int (1-u^2)^2\,du = \frac{1}{64}\int 1-2u^2+u^4\,du$$

$$= \frac{1}{64}\left(u - \frac{2}{3}u^3 + \frac{u^5}{5}\right) + C$$

$$= \frac{1}{64}\left(\sin\theta - \frac{2}{3}\sin^3\theta + \frac{\sin^5\theta}{5}\right) + C$$

$$= \boxed{\frac{1}{64}\left(\frac{x}{\sqrt{x^2+4}} - \frac{2}{3}\left(\frac{x}{\sqrt{x^2+4}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{x^2+4}}\right)^5\right) + C}$$

2(b) $$\int x\arctan x\,dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx$$ slip-in, slip-out

IBP

$$\boxed{\begin{array}{ll} u = \arctan x & dv = x\,dx \\ du = \frac{1}{1+x^2}\,dx & v = \frac{x^2}{2} \end{array}}$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx$$ split-split

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{\cancel{x^2+1}}{\cancel{x^2+1}}^{\,1} - \frac{1}{x^2+1}\,dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\left(x - \arctan x\right) + C$$

$$= \boxed{\frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x + C}$$

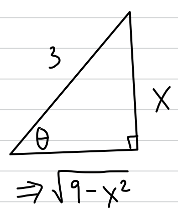
2(c) $\displaystyle\int\sqrt{9-x^2}\,dx = \int\sqrt{\underbrace{9-9\sin^2\theta}_{9(1-\sin^2\theta)\,=\,9\cos^2\theta}}\cdot 3\cos\theta\,d\theta = \int\overbrace{\cancel{\sqrt{9\cos^2\theta}}}^{3\cos\theta}\cdot 3\cos\theta\,d\theta$

Trig Sub

$$\boxed{\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta\,d\theta \end{aligned}}$$

$\hookrightarrow \sin\theta = \dfrac{x}{3}$

$\hookrightarrow \theta = \arcsin\left(\dfrac{x}{3}\right)$



$$= 9\int \cos^2\theta\,d\theta$$

$$= 9\int \frac{1+\cos(2\theta)}{2}\,d\theta$$ Half-Angle Identity

$$= \frac{9}{2}\int 1+\cos(2\theta)\,d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{\overbrace{\cancel{\sin(2\theta)}}^{\cancel{2}\sin\theta\cos\theta}}{\cancel{2}}\right) + C$$ Double Angle Identity

$$= \boxed{\frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) + \underbrace{\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)}_{9}\right) + C}$$

OR//

$$= \boxed{\frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C}$$

2(d) $\displaystyle\int_{\ln 2}^{\ln 5}\frac{2e^x}{e^{2x}-1}\,dx = \int_{\ln 2}^{\ln 5}\frac{2e^x}{(e^x)^2-1}\,dx = \int_2^5 \frac{2}{u^2-1}\,du = \int_2^5\frac{2}{(u-1)(u+1)}\,du$

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x dx \end{aligned}} \qquad \boxed{\begin{aligned} x = \ln 2 &\Rightarrow u = e^{\ln 2} = 2 \\ x = \ln 5 &\Rightarrow u = e^{\ln 5} = 5 \end{aligned}}$$

$$= \int_2^5 \frac{1}{u-1} - \frac{1}{u+1}\,du$$

$$= \ln|u-1| - \ln|u+1| \Big|_2^5$$

PFD

$$\cancel{(u-1)(u+1)}\left(\frac{2}{\cancel{(u-1)(u+1)}} = \frac{A}{u-1} + \frac{B}{u+1}\right)(u-1)(u+1)$$

$2 = A(u+1) + B(u-1)$

$= Au + A + Bu - B$

$= (A+B)u + A - B$

Conditions:

- $A+B=0 \Rightarrow B=-A$
- $A-B=2$

$A-(-A)=2$

$2A=2$

$A=1 \Rightarrow B=-1$

$$= \ln 4 - \ln 6 - (\overset{0}{\cancel{\ln 1}} - \ln 3)$$

$$= \ln\left(\frac{4}{6}\right) + \ln 3$$

$$= \ln\left(\frac{4}{\cancel{6}_{2}}\cdot\cancel{3}\right)$$

$$= \ln\left(\frac{4}{2}\right)$$

$$= \ln 2$$ Match!

3(a) $$\int_{-1}^{6} \frac{15-x}{x^2-6x-7}\,dx = \int_{-1}^{6}\frac{15-x}{(x-7)(x+1)}\,dx = \lim_{t\to -1^+}\int_t^6 \frac{15-x}{(x-7)(x+1)}\,dx$$

Free PFD

$$= \lim_{t\to-1^+}\int_t^6 \frac{1}{x-7}-\frac{2}{x+1}\,dx = \lim_{t\to -1^+} \ln|x-7| - 2\ln|x+1| \Big|_t^6$$

$$= \lim_{t\to-1^+} \cancelto{0}{\ln|\cancelto{1}{-1}|} - \underbrace{2\ln 7}_{\text{Finite}} - \Big(\underbrace{\cancelto{\ln 8}{\ln|t-7|}}_{\text{Finite}} - \cancelto{-\infty}{2\ln|\cancelto{0^+}{t+1}|}\Big) = -\infty$$

Diverges

3(b) $$\int_{-\infty}^{6} \frac{1}{x^2-6x+12}\,dx = \lim_{t\to-\infty}\int_t^6 \frac{1}{x^2-6x+12}\,dx = \lim_{t\to-\infty}\int_t^6\frac{1}{(x-3)^2+3}\,dx$$

Complete the Square

$(x-3)^2 = x^2-6x+9$

$$=\lim_{t\to-\infty}\int_{t-3}^{3} \frac{1}{u^2+3}\,du = \lim_{t\to-\infty} \frac{1}{\sqrt3}\arctan\left(\frac{u}{\sqrt3}\right)\Big|_{t-3}^{3}$$

$$=\lim_{t\to-\infty} \frac{1}{\sqrt3}\left(\cancelto{\frac{\pi}{3}}{\arctan\left(\cancelto{\sqrt3}{\frac{3}{\sqrt3}}\right)} - \cancelto{-\frac{\pi}{2}}{\arctan\left(\cancelto{-\infty}{\frac{t-3}{\sqrt3}}\right)}\right)$$

$$=\frac{1}{\sqrt3}\left(\frac{\pi}{3}+\frac{\pi}{2}\right) = \frac{1}{\sqrt3}\left(\frac{2\pi}{6}+\frac{3\pi}{6}\right) = \frac{5\pi}{6\sqrt3}$$

Converges

3(c) $$\int_0^e x^2\cdot \ln x\,dx = \lim_{t\to0^+}\int_t^e x^2\cdot\ln x\,dx = \lim_{t\to0^+} \frac{x^3}{3}\cdot\ln x\Big|_t^e - \frac13\int_t^e \frac{\cancelto{x^2}{x^3}}{\cancel{x}}\,dx$$

IBP

$$\boxed{\begin{array}{ll} u=\ln x & dv = x^2\,dx \\ du=\frac{1}{x}\,dx & v=\frac{x^3}{3} \end{array}}$$

$$=\lim_{t\to0^+} \frac{x^3}{3}\ln x\Big|_t^e - \frac{x^3}{9}\Big|_t^e$$

$$=\lim_{t\to0^+} \frac{e^3}{3}\cdot\cancelto{1}{\ln e} - \underbrace{\cancelto{0}{\frac{t^3}{3}\cdot\ln t}}_{(*)\ \text{L'H}} - \left(\frac{e^3}{9}-\cancelto{0}{\frac{t^3}{9}}\right) = \underbrace{\frac{e^3}{3}}_{\frac{3e^3}{9}}-\frac{e^3}{9} = \frac{2e^3}{9}$$

$0\cdot(-\infty)$

Converges

(∗) $$\lim_{t\to0^+} \underbrace{t^3\cdot\ln t}_{0\cdot(-\infty)} = \lim_{t\to0^+} \frac{\ln t}{\frac{1}{t^3}} \overset{\text{L'H}}{=} \lim_{t\to0^+} \frac{\frac1t}{\frac{-3}{t^4}} = \lim_{t\to0^+} \frac{-t^4}{3t} = \lim_{t\to0^+} \frac{-\cancelto{0}{t^3}}{3} = 0$$

flip

$\frac{-\infty}{\infty}$

$\frac{-t^4}{3}$

3(d) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{x\ln x}\,dx = \lim_{t\to 0^+}\int_t^{\frac{1}{2}} \frac{1}{x\cdot \ln x}\,dx = \lim_{t\to 0^+}\int_{\ln t}^{\ln\left(\frac{1}{2}\right)} \frac{1}{u}\,du$

$\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}}$ $\boxed{\begin{aligned} x = t &\Rightarrow u = \ln t \\ x = \tfrac{1}{2} &\Rightarrow u = \ln\left(\tfrac{1}{2}\right) \end{aligned}}$

$$= \lim_{t\to 0^+} \ln|u| \Big|_{\ln t}^{\ln\left(\frac{1}{2}\right)}$$

$$= \lim_{t\to 0^+} \underbrace{\ln\left|\ln\left(\tfrac{1}{2}\right)\right|}_{\text{Finite}} - \cancel{\ln\left|\ln t\right|} = -\infty$$

$t \to 0^+$, $\ln t \to -\infty$, $|-\infty| \to \infty$

Diverges

4(a) $-\frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \dots = 4\left(-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = 4\left(\left(\cancel{\arctan 1}\right) - 1\right)$

Missing 1

$\arctan 1 \to \frac{\pi}{4}$

$$= \cancel{4}\left(\frac{\pi}{\cancel{4}}\right) - 4 = \pi - 4$$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$\arctan x = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

4(b) $\frac{1}{e} - \frac{1}{2e^2} + \frac{1}{3e^3} - \frac{1}{4e^4} + \frac{1}{5e^5} - \frac{1}{6e^6} + \dots = \frac{1}{e} - \frac{\left(\frac{1}{e}\right)^2}{2} + \frac{\left(\frac{1}{e}\right)^3}{3} - \frac{\left(\frac{1}{e}\right)^4}{4} + \dots$

$$= \boxed{\ln\left(1 + \frac{1}{e}\right)}$$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

4(c) $2 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \dots = 1 + 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \dots$

Extra

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$$= 1 + \cancel{\cos(\pi)} = 1 - 1 = 0$$

$\cos(\pi) \to -1$

4(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n+1)!} \cdot \frac{\frac{\pi}{3}}{\frac{\pi}{3}} = \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}$

$9^n = 3^{2n}$

$\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$= \frac{3}{\pi}\cancel{\sin\left(\frac{\pi}{3}\right)} = \frac{3\sqrt{3}}{2\pi}$$

$\sin\left(\frac{\pi}{3}\right) \to \frac{\sqrt{3}}{2}$

4(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, 2^{n+1} (\ln 3)^n}{3!\; n!} = \frac{-2}{3!} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (\ln 3)^n}{n!} = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{(-2\ln 3)^n}{n!}$

constant

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= -\frac{1}{3} e^{-2\ln 3} = -\frac{1}{3} \cancel{e}^{\cancel{\ln}(3^{-2})} = -\frac{1}{3} \cdot \frac{1}{9} = \frac{-1}{27}$$

$\frac{1}{3^2}$

4(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n - 1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$

split-split

Both Geometric Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$= \frac{1}{1-\left(-\frac{2}{3}\right)} - \frac{1}{1-\frac{1}{3}} = \frac{3}{5} - \frac{3}{2} = \frac{6}{10} - \frac{15}{10} = \frac{-9}{10}$$

$+$ $\frac{5}{3}$ $\frac{2}{3}$

Match!

5(a) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{(n+6)^6} + \frac{1}{6^n} = \sum_{n=1}^{\infty} \frac{6}{(n+6)^6} + \sum_{n=1}^{\infty} \frac{1}{6^n}$

$$6\sum_{n=1}^{\infty} \frac{1}{(n+6)^6} \approx 6 \sum \frac{1}{n^6}$$

Converges by GST

with $|r| = \left|\frac{1}{6}\right| = \frac{1}{6} < 1$

Constant Multiple of Convergent p-Series $p = 6 > 1$ is Convergent

Bound Terms

$$\frac{6}{(n+6)^6} \le \frac{6}{n^6}$$

$\Rightarrow$ Series also Converges by CT

Sum of 2 Convergent Series is Convergent

O.S. Converges

5(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^6}{\ln n}$ Diverges by nTDT because

$$\lim_{n\to\infty} \frac{n^6}{\ln n} = \lim_{x\to\infty} \frac{x^6}{\ln x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{6x^5}{\frac{1}{x}} = \lim_{x\to\infty} 6x^6 = \infty \neq 0$$

6(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^3+7}{n^7+3} \xrightarrow{A.S.} \sum_{n=1}^{\infty}\frac{n^3+7}{n^7+3} \approx \sum_{n=1}^{\infty}\frac{n^3}{n^7} = \sum_{n=1}^{\infty}\frac{1}{n^4}$ Converges p-Series $p=4>1$

$$\lim_{n\to\infty}\frac{\frac{n^3+7}{n^7+3}}{\frac{1}{n^4}} = \lim_{n\to\infty}\frac{n^7+7n^4}{n^7+3}\cdot\frac{\frac{1}{n^7}}{\frac{1}{n^7}} = \lim_{n\to\infty}\frac{1+\cancel{\frac{7}{n^3}}^{\,0}}{1+\cancel{\frac{3}{n^7}}^{\,0}} = 1$$

Finite Non-zero

$\Rightarrow$ A.S. also Converges by LCT

$\Rightarrow$ O.S. Absolutely Convergent by Definition

6(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \xrightarrow{A.S.} \sum_{n=1}^{\infty}\frac{1}{n}$ A.S. Diverges Harmonic p-Series $p=1$

AST

1 $b_n=\frac{1}{n}>0$

2 $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$

3. Terms Decreasing

$b_{n+1}=\frac{1}{n+1}\le\frac{1}{n}=b_n$

OR// $f(x)=\frac{1}{x} \Rightarrow f'(x)=\frac{-1}{x^2}<0$

O.S. Converges by AST

O.S. Conditionally Convergent by Definition

7(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(3x+2)^n}{(n+7)^2\cdot 4^n}$

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(3x+2)^{n+1}}{(n+8)^2\,4^{n+1}}}{\frac{\cancel{(-1)^n}(3x+2)^n}{(n+7)^2\cdot 4^n}}\right| = \lim_{n\to\infty}\left|\frac{(3x+2)^{n+1}}{(3x+2)^n}\right|\cdot\frac{(n+7)^2}{(n+8)^2}\cdot\frac{4^n}{4^{n+1}}$$

$|3x+2|$ $\quad \left(\frac{n+7}{n+8}\right)^2 \to 1$ $\quad 4$

$$= \lim_{n\to\infty}\frac{|3x+2|}{4} = \frac{|3x+2|}{4} < 1$$

Converges by Ratio Test when

$|3x+2|<4$ $\qquad -4<3x+2<4$ $\qquad -2<x<\frac{2}{3}$

$-2 \quad -2 \quad -2$

$-6<3x<2$

3

Manually Check Convergence at End points

Take $x=-2$. Series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n\left(3(-2)+2\right)^n}{(n+7)^2\cdot 4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n\overbrace{(-4)^n}^{(-1)^n\cdot\cancel{4^n}}}{(n+7)^2\cdot \cancel{4^n}}=\sum_{n=1}^{\infty}\frac{1}{(n+7)^2}\approx\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$\cancel{(-1)}^{2n}$ Even $\to 1$

Converges p-Series $p=2>1$

Bound Terms

$$\frac{1}{(n+7)^2}\leq\frac{1}{n^2}\quad\Rightarrow \text{Series also Converges by CT}$$

(OR, LCT works to, use Limit)

Take $x=\frac{2}{3}$. Series becomes

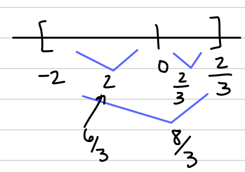
$$\sum_{n=1}^{\infty}\frac{(-1)^n\left(\cancel{3}\left(\frac{2}{\cancel{3}}\right)+2\right)^n}{(n+7)^2\cdot 4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n\,\cancel{4^n}}{(n+7)^2\cdot\cancel{4^n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+7)^2}\xrightarrow{\text{A.S.}}\sum_{n=1}^{\infty}\frac{1}{(n+7)^2}$$

A.S. Converges as shown above

$\Rightarrow$ original Alternating Series Converges by A.C.T.

OR, Can use AST on Original Series

Finally, $I=\left[-2,\frac{2}{3}\right]$ and $R=\frac{4}{3}$



Half $\frac{4}{3}$

7(b) $\sum_{n=1}^{\infty} n^n (x-7)^n$

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\overbrace{(n+1)^{n+1}}^{(n+1)^n(n+1)}(x-7)^{n+1}}{n^n\,(x-7)^n}\right|=\lim_{n\to\infty}\underbrace{\frac{(n+1)^n}{n^n}}_{e}\cdot\underbrace{(n+1)}_{\infty}|x-7|=\infty>1$$

Diverges by Ratio Test

Unless $x-7=0\Rightarrow x=7$

Finally, $I=\{7\}$

$R=0$

7(c) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{\cancel{(-1)^n} x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{x^{2n+1}}\right| \cdot \frac{\cancel{(2n+1)!}}{(2n+3)!}$$

$|x|^2$

$(2n+3)(2n+2)\cancel{(2n+1)!}$

$$= \lim_{n\to\infty} \frac{|x|^2 \to 0}{(2n+3)(2n+2) \to \infty} = 0 < 1 \quad \text{Converges by Ratio Test for all } x$$

Finally, $I = (-\infty, \infty)$ $R = \infty$

8(a) $\frac{d}{dx}\left(6x^3 \arctan(6x)\right) = \frac{d}{dx}\left(6x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (6x)^{2n+1}}{2n+1}\right) = \frac{d}{dx}\left(6x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n+1} x^{2n+1}}{2n+1}\right)$

Multiple won't change Convergence

Need $|6x| < 1$

$\hookrightarrow |x| < \frac{1}{6}$

$\hookrightarrow R = 1$

$$= \frac{d}{dx}\left(\underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n+2} x^{2n+4}}{2n+1}}_{R=\frac{1}{6}}\right)$$

Note: Q did not ask for Radius.

$$= \underbrace{\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n+2} (2n+4) x^{2n+3}}{2n+1}}}_{R=\frac{1}{6} \text{ STILL after Differentiation.}}$$

8(b) $\int_0^1 x^3 \cos(x^3)\,dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}\,dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}\,dx$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(2n)!\,(6n+4)} \Bigg|_0^1$$

$$= \frac{x^4}{1\cdot 4} - \frac{x^{10}}{2!\cdot(10)} + \frac{x^{16}}{4!\,(16)} - \cdots \Bigg|_0^1$$

$$\begin{array}{r} \overset{2}{2}4 \quad 4! \\ \underline{16} \\ 144 \\ \underline{240} \\ 384 \end{array}$$

Estimate

$$= \frac{1}{4} - \frac{1}{20} + \frac{1}{384} - \cdots - (0 - 0 + 0 - \cdots) \approx \frac{1}{4} - \frac{1}{20} = \frac{4}{20} = \frac{1}{5}$$

Using A.S.E.T. we can Estimate the Full Sum using only the first Two Terms with Error at most $|\text{First Neglected Term}| = \frac{1}{384} < \frac{1}{200}$ as desired.

9(a) $\frac{1}{(1+7x)^2} = \overset{\text{2nd}}{\frac{d}{dx}}\left(\frac{-1}{7(1+7x)}\right)^{\text{1st P.S.}} = \frac{d}{dx}\left(\frac{-1}{7(1-(-7x))}\right) = \frac{d}{dx}\left(-\frac{1}{7}\sum_{n=0}^{\infty}(-7x)^n\right)$

Need $|-7x|<1$

$\Rightarrow |x|<\frac{1}{7}$

$\Rightarrow R=\frac{1}{7}$

$= \frac{d}{dx}\left(-\frac{1}{7}\sum_{n=0}^{\infty}(-1)^n 7^n x^n\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^{n+1}7^{n-1}x^n\right)$

$= \boxed{\sum_{n=0}^{\infty}(-1)^{n+1}7^{n-1}\cdot n x^{n-1}}$ $\boxed{R=\frac{1}{7}}$ STILL after Differentiation

Need $\left|\frac{-x}{3}\right|<1 \Rightarrow |x|<3$

$\Rightarrow R=3$

9(b) $\ln(3+x) = \int \frac{1}{3+x}\,dx = \int \frac{1}{3(1+x/3)}\,dx = \int \frac{1}{3\left(1-\left(-\frac{x}{3}\right)\right)}\,dx = \int \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n dx$

$= \int \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^n}\,dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^{n+1}}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)} + C$

R=3 after Integration

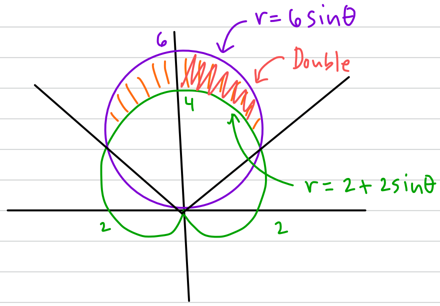
$= \frac{x}{3\cdot 1} - \frac{x^2}{3^2\cdot 2} + \frac{x^3}{3^3\cdot 3} - \cdots + C$ Expand to Confirm Every term has an x.

Test $x=0$ ← Center Point, definitely in Domain of Series

$\overset{\ln 3}{\cancel{\ln(3+0)}} = 0-0+0-\cdots+C \Rightarrow C=\ln 3$

Finally, $\ln(3+x) = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)} + \ln 3}$

10(a)



Intersect?

$2+2\sin\theta = 6\sin\theta$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

↪ $\theta = \pi/6,\ 5\pi/6$

$\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (\text{Outer Radius})^2 - (\text{Inner Radius})^2\,d\theta$

$= \boxed{\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (6\sin\theta)^2 - (2+2\sin\theta)^2\,d\theta}$

Do Not Evaluate

Symmetry:

$$\boxed{= \cancel{2}\left(\cancel{\frac{1}{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (6\sin\theta)^2 - (2+2\sin\theta)^2\, d\theta\right)}$$

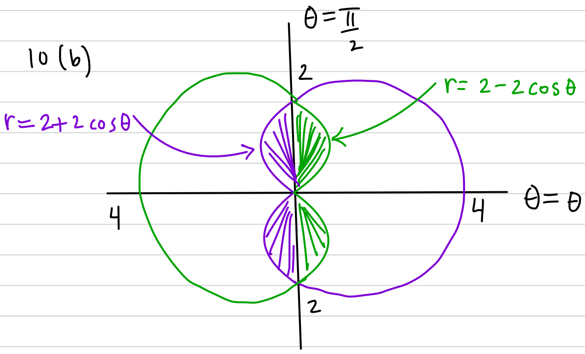
Double using Symmetry

OR//

$$\boxed{= \cancel{2}\left(\cancel{\frac{1}{2}}\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (6\sin\theta)^2 - (2+2\sin\theta)^2\, d\theta\right)}$$

Double using Symmetry

10 (b)



$$\text{Area} = \boxed{4\left(\frac{1}{2}\int_{0}^{\frac{\pi}{2}} (\text{Radius})^2\, d\theta\right)}$$

Quadruple using Symmetry

$$= \boxed{4\left(\frac{1}{2}\int_{0}^{\frac{\pi}{2}} (2-2\cos\theta)^2\, d\theta\right)}$$

More ↓ ⋮

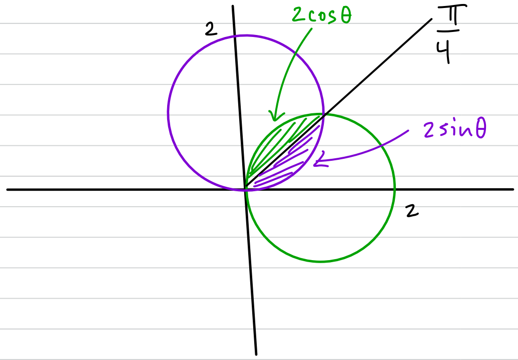
OR// $= 4\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (2+2\cos\theta)^2\, d\theta\right)$ OR// $= 4\left(\frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} (2+2\cos\theta)^2\, d\theta\right)$

OR// $= 4\left(\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi} (2-2\cos\theta)^2\, d\theta\right)$

OR// $= 2\left(\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2-2\cos\theta)^2\, d\theta\right)$ OR// $= 2\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2+2\cos\theta)^2\, d\theta\right)$

Double using Symmetry

10(c)



Intersect ?

$\sin\theta = \cos\theta$

$\hookrightarrow \theta = \frac{\pi}{4}$

$$\text{Area} = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{4}} (\text{Polar Radius})^2 \, d\theta\right)$$

Double using Symmetry

$$\boxed{= 2\left(\frac{1}{2}\int_0^{\frac{\pi}{4}} (2\sin\theta)^2 \, d\theta\right)}$$

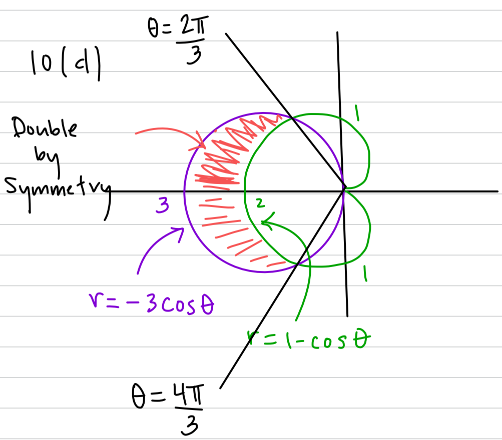
OR//

$$\boxed{= 2\left(\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos\theta)^2 \, d\theta\right)}$$

OR//

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (2\sin\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos\theta)^2 d\theta$$

10(d)



Intersect ?

$1-\cos\theta = -3\cos\theta$

$1 = -2\cos\theta$

$\cos\theta = -\frac{1}{2}$ $\qquad$ $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ by symmetry

$$\text{Area} = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\text{Outer Radius})^2 - (\text{Inner Radius})^2 \, d\theta$$

$$\boxed{= \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (-3\cos\theta)^2 - (1-\cos\theta)^2 \, d\theta}$$

OR//

$$= \boxed{2\left(\frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (-3\cos\theta)^2 - (1-\cos\theta)^2 \, d\theta\right)}$$

Double by Symmetry

OR//

$$= 2\left(\frac{1}{2}\int_{\pi}^{\frac{4\pi}{3}} (-3\cos\theta)^2 - (1-\cos\theta)^2 \, d\theta\right)$$