Math 121 Self-Assessment Practice Worksheet #6 Answer Key

Compute each of the following Integrals.

Note: Since these are Improper Type II integrals, with Vertical asymptotes, be careful to check *from which side* you are approaching or sneak-attacking the undefined value. In some cases, it may change the final value.

1. $\displaystyle\int_{-4}^{-3} \frac{8-x}{x^2+2x-8}\,dx$ $\displaystyle\int_{-4}^{-3} \frac{8-x}{(x+4)(x-2)}\,dx$

$$= \lim_{t\to -4^+} \int_{t}^{-3} \frac{8-x}{(x+4)(x-2)}\,dx$$

$$= \lim_{t\to -4^+} \int_{t}^{-3} \frac{1}{x-2} - \frac{2}{x+4}\,dx$$

$$= \lim_{t\to -4^+} \ln|x-2| - 2\ln|x+4|\Big|_{t}^{-3}$$

$$= \lim_{t\to -4^+} \ln|-5| - 2\overset{0}{\cancel{\ln 1}} - \left(\overset{\ln 6}{\cancel{\ln|t-2|}} - 2\overset{-\infty}{\cancel{\ln\left|\overset{0^+}{\cancel{\overset{-4^+}{\cancel{t}}+4}}\right|}} \right)$$

$$= \underbrace{\ln 5 - \ln 6}_{\text{finite}} - \infty = \boxed{-\infty} \quad \text{Integral Diverges}$$

Partial Fractions Decomposition:

$$\frac{8-x}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}$$

Clearing the denominator yields:

$8-x = A(x-2) + B(x+4)$
$8-x = (A+B)x - 2A + 4B$
so that $A+B=-1$, and $-2A+4B=8$
Solve for $A=-2$ and $B=1$

$$2.\ \int_0^e \ln x \ dx = \lim_{t\to 0^+} \int_t^e \ln x \cdot 1 \ dx$$

$$= \lim_{t\to 0^+} x\ln x \Big|_t^e - \int_t^e 1 \ dx$$

$$= \lim_{t\to 0^+} x\ln x \Big|_t^e - x\Big|_t^e$$

$$= \lim_{t\to 0^+} e\cancelto{1}{\ln e} - \cancelto{0 \text{ see } (*)}{t\ln t}^{0\cdot(-\infty)} - (e - \cancelto{0}{t})$$

$$\overset{\text{L'H }*}{=} e - 0 - e = \boxed{0} \quad \text{Integral Converges}$$

IBP: $\boxed{\begin{array}{ll} u = \ln x & dv = 1dx \\ du = \frac{1}{x}dx & v = x \end{array}}$

$$(*)\ \lim_{t\to 0^+} t\ln t^{0\cdot(-\infty)} = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}}^{\frac{-\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0^+} \frac{1}{t}\cdot\left(-\frac{t^2}{1}\right) = \lim_{t\to 0^+} -\cancelto{0}{t} = 0$$

$$3.\ \int_0^1 \frac{e^{\frac{1}{x}}}{x^2} \ dx = \lim_{t\to 0^+} \int_t^1 \frac{e^{\frac{1}{x}}}{x^2} \ dx = -\lim_{t\to 0^+} \int_{\frac{1}{t}}^1 e^u \ du$$

$$= \lim_{t\to 0^+} -e^u\Big|_{\frac{1}{t}}^1 = \lim_{t\to 0^+} -e + e^{\cancelto{\infty}{\frac{1}{t}}} = \boxed{\infty} \quad \text{Integral Diverges}$$

$$\boxed{\begin{array}{rl} u & = \frac{1}{x} \\ du & = -\frac{1}{x^2} \ dx \\ -du & = \frac{1}{x^2} \ dx \end{array}} \qquad \boxed{\begin{array}{l} x = t \Rightarrow u = \frac{1}{t} \\ x = 1 \Rightarrow u = 1 \end{array}}$$

$$4.\quad \int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^{\frac{1}{x}}}{x^2}\,dx = -\lim_{t\to 0^-} \int_{-1}^{\frac{1}{t}} e^u\,du$$

$$= \lim_{t\to 0^-} -e^u\Big|_{-1}^{\frac{1}{t}} = \lim_{t\to 0^-} -\overset{0}{\cancel{e^{\overset{-\infty}{\cancel{\frac{1}{t}}}}}} + e^{-1} = 0 + e^{-1} = \boxed{\frac{1}{e}} \quad \text{Integral Converges}$$

Same u-sub as in 3 above, different improper limit finish

$$\boxed{\begin{aligned} u &= \frac{1}{x} \\ du &= -\frac{1}{x^2}\,dx \\ -du &= \frac{1}{x^2}\,dx \end{aligned}} \qquad \boxed{\begin{aligned} &x = -1 \Rightarrow u = -1 \\ &x = t \Rightarrow u = \frac{1}{t} \end{aligned}}$$

$$5.\quad \int_{0}^{\frac{1}{2}} \frac{1}{x\ln x}\,dx \quad \lim_{t\to 0^+} \int_{t}^{\frac{1}{2}} \frac{1}{x\ln x}\,dx$$

$$= \lim_{t\to 0^+} \int_{\ln t}^{\ln\frac{1}{2}} \frac{1}{u}\,du$$

$$= \lim_{t\to 0^+} \ln|u|\Big|_{\ln t}^{\ln\frac{1}{2}}$$

$$= \lim_{t\to 0^+} \underbrace{\ln|\ln\frac{1}{2}|}_{\text{finite}} - \overset{\infty}{\cancel{\ln\overset{|-\infty|}{|\ln t|}}} = \boxed{-\infty} \quad \text{Integral Diverges}$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}} \qquad \boxed{\begin{aligned} &x = t \Rightarrow u = \ln t \\ &x = \tfrac{1}{2} \Rightarrow u = \ln\frac{1}{2} \end{aligned}}$$

6. $$\int_1^2 \frac{1}{x\ln x}\,dx = \lim_{t\to 1^+}\int_t^2 \frac{1}{x\ln x}\,dx$$

$$= \lim_{t\to 1^+}\int_t^2 \frac{1}{u}\,du$$

$$= \lim_{t\to 1^+} \ln|u|\Big|_{\ln t}^{\ln 2}$$

$$= \lim_{t\to 1^+} \underbrace{\ln|\ln 2|}_{\text{finite}} - \overset{-\infty}{\cancel{\ln\left|\overset{0^+}{\cancel{\ln \overset{1^+}{\cancel{t}}}}\right|}} = -(-\infty) = \boxed{\infty} \quad \text{Integral Diverges}$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}} \qquad \boxed{\begin{aligned} &x = t \Rightarrow u = \ln t \\ &x = 2 \Rightarrow u = \ln 2 \end{aligned}}$$

7. $$\int_0^e x^2 \ln\left(x^2\right)\,dx = \lim_{t\to 0^+}\int_t^e x^2\ln\left(x^2\right)\,dx$$

$$= \lim_{t\to 0^+} \frac{x^3}{3}\cdot\ln(x^2)\Big|_t^e - \frac{2}{3}\int_t^e x^2\,dx$$

$$= \lim_{t\to 0^+} \frac{x^3}{3}\cdot\ln(x^2)\Big|_t^e - \frac{2}{9}x^3\Big|_t^e$$

$$= \lim_{t\to 0^+} \frac{e^3}{3}\cdot\overset{2}{\cancel{\ln(e^2)}} - \overset{0 \text{ see } (*)}{\cancel{\frac{t^3}{3}\cdot\ln(t^2)}}^{0\cdot(-\infty)} - \left(\frac{2}{9}e^3 - \frac{2}{9}\overset{0}{\cancel{t^3}}\right)$$

$$\overset{\text{L'H}}{=}{}^{*}\ \frac{2e^3}{3} - 0 - \frac{2e^3}{9} = \frac{6e^3}{9} - \frac{2e^3}{9} = \boxed{\frac{4e^3}{9}} \quad \text{Integral Converges}$$

IBP: $$\boxed{\begin{aligned} &u = \ln(x^2) & &dv = x^2dx \\ &du = \frac{1}{x^2}(2x)dx = \frac{2}{x}dx & &v = \frac{x^3}{3} \end{aligned}}$$

$$(*)\ \lim_{t\to0^+} t^3\ln(t^2)^{0\cdot(-\infty)} = \lim_{t\to0^+}\frac{\ln(t^2)}{\dfrac{1}{t^3}} \overset{\frac{-\infty}{\infty}}{\underset{}{\overset{\text{L'H}}{=}}} \lim_{t\to0^+}\frac{\dfrac{1}{t^2}(2t)}{-\dfrac{3}{t^4}} = \lim_{t\to0^+}\frac{2}{t}\cdot\left(-\frac{t^4}{3}\right) = \lim_{t\to0^+} -\frac{2\cancel{t^3}^{\,0}}{\cancel{3}} = 0$$

Note: You can also simplify the original integral slightly by using log algebra to pull down the power of 2 inside the log, that is:

$$\int_0^e x^2\ln\left(x^2\right)\ dx = 2\int_0^e x^2\ln x\ dx$$

It is still an IBP with a L'H finish, etc.

8. $$\int_0^{e^5}\frac{1}{x\left(25+(\ln x)^2\right)}\ dx = \lim_{t\to0^+}\int_t^{e^5}\frac{1}{x(25+(\ln x)^2)}\ dx$$

$$= \lim_{t\to0^+}\int_{\ln t}^{5}\frac{1}{25+w^2}\ dw$$

$$= \lim_{t\to0^+}\frac{1}{5}\arctan\left(\frac{w}{5}\right)\Big|_{\ln t}^{5}$$

$$= \lim_{t\to0^+}\frac{1}{5}\left(\arctan\left(\frac{\cancel{5}}{\cancel{5}}\right) - \arctan\left(\frac{\ln t}{5}\right)\right)$$

$$= \lim_{t\to0^+}\frac{1}{5}\left(\arctan(1) - \arctan\left(\frac{\ln t^{\,0^+}}{5}\right)^{-\infty}\right)^{-\frac{\pi}{2}}$$

$$= \frac{1}{5}\left(\frac{\pi}{4} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{5}\left(\frac{3\pi}{4}\right)$$

$$= \boxed{\frac{3\pi}{20}}\quad \text{Integral Converges}$$

Note: $\ln t$ approaches $-\infty$ as t approaches 0 positively. Then $\arctan\ln t$ approaches $-\dfrac{\pi}{2}$ as the input $\ln t$ approaches $-\infty$.

$$\boxed{\begin{aligned} w &= \ln x \\ dw &= \frac{1}{x}\,dx \end{aligned}} \quad \boxed{\begin{aligned} &x = t \Rightarrow w = \ln t \\ &x = e^5 \Rightarrow w = \ln(e^5) = 5 \end{aligned}}$$

Determine whether each of the following Sequences Converge or Diverge.

First, recall that given a Sequence, $\{a_n\}_{n=1}^{\infty}$, study the Limiting value of the terms a_n, to determine whether the Sequence Converges or Diverges. That is, if $\lim\limits_{n\to\infty} a_n$ is finite, then the Sequence Converges, and otherwise it Diverges. Be careful not to mix up Sequence limit work with Infinite Series Convergence Tests.

Key: Remember if you need to apply L'Hôpital's Rule, then you must switch to studying the Related Function in x. IF you are only using an algebra method, then you can continue to work in the variable n

9. $\left\{\dfrac{3n^7 - 2n + 1}{8n^7 + 9}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{3n^7 - 2n + 1}{8n^7 + 9} = \lim_{n\to\infty} \frac{3n^7 - 2n + 1}{8n^7 + 9} \cdot \frac{\frac{1}{n^7}}{\frac{1}{n^7}} = \lim_{n\to\infty} \frac{3 - \overset{0}{\cancel{\frac{2}{n^6}}} + \overset{0}{\cancel{\frac{1}{n^7}}}}{8 + \overset{0}{\cancel{\frac{9}{n^7}}}} = \boxed{\frac{3}{8}} \quad \text{Sequence Converges}$$

10. $\left\{n^2 \sin\left(\dfrac{1}{n^2}\right)\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} n^2 \sin\left(\frac{1}{n^2}\right) = \lim_{x\to\infty} x^2 \sin\left(\frac{1}{x^2}\right)^{\infty\cdot 0} = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}}^{\frac{0}{0}}$$

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x^2}\right) \cdot \left(-\cancel{\frac{2}{x^3}}\right)}{-\cancel{\frac{2}{x^3}}} = \lim_{x\to\infty} \cos\left(\overset{0}{\cancel{\frac{1}{x^2}}}\right) = \boxed{1} \quad \text{Sequence Converges}$$

11. $\left\{\arctan\left(\dfrac{n^7+7}{\sqrt{3}n^7+1}\right)\right\}_{n=1}^{\infty}$

$$\begin{aligned}
\lim_{n\to\infty} \arctan\left(\frac{n^7+7}{\sqrt{3}n^7+1}\right) &= \arctan\left(\lim_{n\to\infty} \frac{n^7+7}{\sqrt{3}n^7+1}\cdot\frac{\frac{1}{n^7}}{\frac{1}{n^7}}\right) \\
&= \arctan\left(\lim_{n\to\infty} \frac{1+\cancelto{0}{\frac{7}{n^7}}}{\sqrt{3}+\cancelto{0}{\frac{1}{n^7}}}\right) \\
&= \arctan\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{\pi}{6}} \quad \text{Sequence Converges}
\end{aligned}$$

12. $\left\{\dfrac{n^7}{\ln n}\right\}_{n=5}^{\infty}$

$$\lim_{n\to\infty} \frac{n^7}{\ln n}^{\frac{\infty}{\infty}} = \lim_{x\to\infty} \frac{x^7}{\ln x}^{\frac{\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{7x^6}{\frac{1}{x}} = \lim_{x\to\infty} 7\cancelto{\infty}{x^7} = \boxed{\infty} \quad \text{Sequence Diverges}$$

13. $\left\{\dfrac{(n+2)!}{(n-1)!}\right\}_{n=1}^{\infty}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+2)!}{(n-1)!} &= \lim_{n\to\infty} \frac{(n+2)(n+1)n\cancel{(n-1)!}}{\cancel{(n-1)!}} \\
&= \lim_{n\to\infty} \cancelto{\infty}{(n+2)(n+1)n} = \boxed{\infty} \quad \text{Sequence Diverges}
\end{aligned}$$

14. $\left\{\dfrac{n^2}{e^n}\right\}_{n=2}^{\infty}$

$$\lim_{n\to\infty} \frac{n^2}{e^n}^{\frac{\infty}{\infty}} = \lim_{x\to\infty} \frac{x^2}{e^n}^{\frac{\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2x}{e^x}^{\frac{\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \cancelto{0}{\frac{2}{\cancelto{\infty}{e^x}}} = \boxed{0} \quad \text{Sequence Converges}$$

15. $\left\{\left(1-\arcsin\left(\frac{5}{n^3}\right)\right)^{n^3}\right\}_{n=1}^{\infty}$

$$
\begin{aligned}
\lim_{n\to\infty}\left(1-\arctan\left(\frac{3}{n^4}\right)\right)^{n^4} &= \lim_{x\to\infty}\left(1-\arctan\left(\frac{3}{x^4}\right)\right)^{x^4} \\
&\overset{1^\infty}{=} e^{\lim\limits_{x\to\infty}\ln\left(\left(1-\arctan\left(\frac{3}{x^4}\right)\right)^{x^4}\right)} \\
&= e^{\lim\limits_{x\to\infty} x^4\ln\left(1-\arctan\left(\frac{3}{x^4}\right)\right)} \\
&\overset{\infty\cdot 0}{=} e^{\lim\limits_{x\to\infty}\frac{\ln\left(1-\arctan\left(\frac{3}{x^4}\right)\right)^{\frac{0}{0}}}{\frac{1}{x^4}}} \\
&\overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\frac{\frac{1}{1-\arctan\left(\cancelto{0}{\frac{3}{x^4}}\right)}\left(-\frac{1}{1+\left(\cancelto{0}{\frac{3}{x^4}}\right)^2}\right)\left(\cancel{-}\frac{12}{\cancel{x^5}}\right)}{\cancel{-}\frac{4}{\cancel{x^5}}}} \\
&= e^{1(-1)(3)} = \boxed{e^{-3}} \quad \text{Sequence Converges}
\end{aligned}
$$

16. $\left\{\left(\dfrac{n+1}{n}\right)^n\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \overset{1^\infty}{=} \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

$$= e^{\lim_{x\to\infty}\ln\left[\left(1+\frac{1}{x}\right)^x\right]} = e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)^{\infty\cdot 0}}$$

$$= e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)^{\frac{0}{0}}}{\frac{1}{x}}} \overset{\text{L'H}}{=} e^{\lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{1}{x}}\right)\left(-\frac{1}{\cancel{x^2}}\right)}{-\frac{1}{\cancel{x^2}}}}$$

$$= e^{\lim_{x\to\infty}\left(\frac{1}{1+\frac{1}{x}}\right)} = e^1 = \boxed{e} \quad \text{Sequence Converges}$$

Memorize this Limit Fact. It will be very helpful in the future.

17. $\left\{\dfrac{(n+7)^9}{(n+6)^9}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty}\frac{(n+7)^9}{(n+6)^9} = \lim_{n\to\infty}\left(\frac{n+7}{n+6}\right)^9 = \lim_{n\to\infty}\left(\frac{n+7}{n+6}\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)\right)^9$$

$$= \lim_{n\to\infty}\left(\frac{1+\frac{7}{n}}{1+\frac{6}{n}}\right)^9 = \boxed{1} \quad \text{Sequence Converges}$$

18. $\left\{\dfrac{(2n-1)!}{(2n+1)!}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty}\frac{(2n-1)!}{(2n+1)!} = \lim_{n\to\infty}\frac{\cancel{(2n-1)!}}{(2n+1)(2n)\cancel{(2n-1)!}}$$

$$= \lim_{n\to\infty}\frac{1}{(2n+1)(2n)} = \frac{1}{\infty} = \boxed{0} \quad \text{Sequence Converges}$$