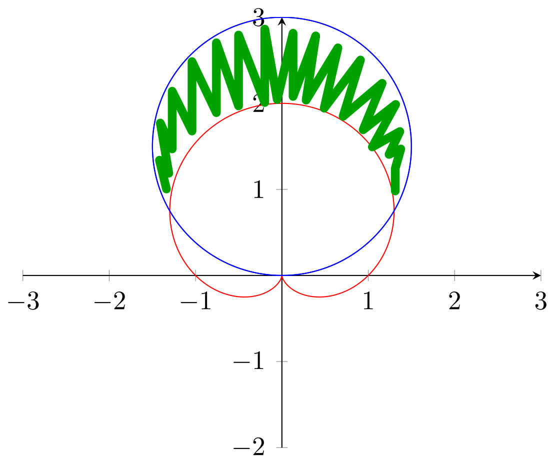
For each of the following, do the following **THREE** things:

(A) Sketch the Polar Curve(s) and **Shade** the described bounded region.

(B) Compute the described bounded area.

(C) Set-Up but **DO NOT EVALUATE** another slightly different integral representing the same area of the described region.

1. Compute the area bounded outside the polar curve $r = 1 + \sin\theta$ and inside the polar curve $r = 3\sin\theta$.



These two polar curves intersect when

$1 + \sin\theta = 3\sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$ or $\theta = \dfrac{5\pi}{6}$.

Using symmetry, we will integrate from $\theta = \dfrac{\pi}{6}$ to $\theta = \dfrac{\pi}{2}$ and double that area.

$$\begin{aligned}
\text{Area} = A &= 2\left(\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\text{outer } \ r)^2 - (\text{inner } \ r)^2 \ d\theta\right) \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3\sin\theta)^2 - (1+\sin\theta)^2 \ d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta) \ d\theta
\end{aligned}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta \ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\sin^2\theta - 1 - 2\sin\theta \ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\left(\frac{1-\cos(2\theta)}{2}\right) - 1 - 2\sin\theta \ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\left(1-\cos(2\theta)\right) - 1 - 2\sin\theta \ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 4\cos(2\theta) - 1 - 2\sin\theta \ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3 - 4\cos(2\theta) - 2\sin\theta \ d\theta$$

$$= 3\theta - 2\sin(2\theta) + 2\cos\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(3\left(\frac{\pi}{2}\right) - 2\sin\left(\frac{2\pi}{2}\right) + 2\cos\left(\frac{\pi}{2}\right)\right) - \left(3\left(\frac{\pi}{6}\right) - 2\sin\left(\frac{2\pi}{6}\right) + 2\cos\left(\frac{\pi}{6}\right)\right)$$

$$= \left(3\left(\frac{\pi}{2}\right) - 2\cancelto{0}{\sin\pi} + 2\cancelto{0}{\cos\left(\frac{\pi}{2}\right)}\right) - \left(\frac{\pi}{2} - 2\cancelto{\frac{\sqrt{3}}{2}}{\sin\left(\frac{\pi}{3}\right)} + 2\cancelto{\frac{\sqrt{3}}{2}}{\cos\left(\frac{\pi}{6}\right)}\right)$$

$$= \frac{3\pi}{2} - 2(0) + 2(0) - \left(\frac{\pi}{2} - 2\cancel{\left(\frac{\sqrt{3}}{2}\right)} + 2\cancel{\left(\frac{\sqrt{3}}{2}\right)}\right) = \frac{3\pi}{2} - \frac{\pi}{2} = \boxed{\pi}$$

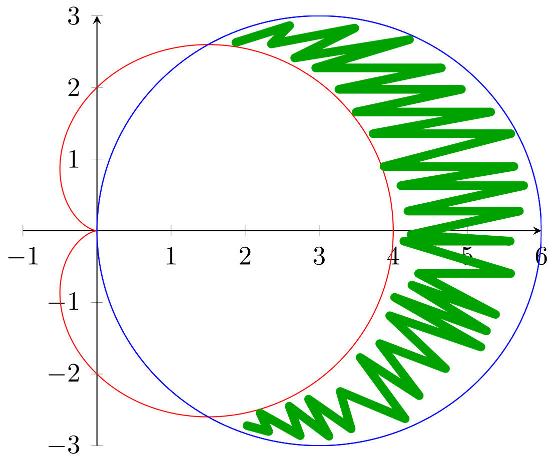
Alternate option:

$$\text{Area}=A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 - (1+\sin\theta)^2 \ d\theta$$

Alternate option:

$$\text{Area}=A = 2\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (3\sin\theta)^2 - (1+\sin\theta)^2 \ d\theta\right)$$

2. Compute the area bounded outside the polar curve $r = 2 + 2\cos\theta$ and inside the polar curve $r = 6\cos\theta$.



These two polar curves intersect when $2 + 2\cos\theta = 6\cos\theta \Rightarrow 4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = -\frac{\pi}{3}$ or $\theta = \frac{\pi}{3}$. Using symmetry, we will integrate from $\theta = 0$ to $\theta = \frac{\pi}{3}$ and double that area.

$$
\begin{aligned}
\text{Area} = A \quad &= \cancel{2}\left(\frac{\cancel{1}}{\cancel{2}}\int_0^{\frac{\pi}{3}} \left((\text{outer} \;\; r)^2 - (\text{inner} \;\; r)^2\right)\, d\theta\right) \\
&= \int_0^{\frac{\pi}{3}} \left((6\cos\theta)^2 - (2 + 2\cos\theta)^2\right)\, d\theta \\
&= \int_0^{\frac{\pi}{3}} 36\cos^2\theta - (4 + 8\cos\theta + 4\cos^2\theta)\; d\theta \\
&= \int_0^{\frac{\pi}{3}} 36\cos^2\theta - 4 - 8\cos\theta - 4\cos^2\theta \; d\theta \\
&= \int_0^{\frac{\pi}{3}} 32\cos^2\theta - 4 - 8\cos\theta \; d\theta \\
&= \int_0^{\frac{\pi}{3}} 32\left(\frac{1 + \cos(2\theta)}{2}\right) - 4 - 8\cos\theta \; d\theta \\
&= \int_0^{\frac{\pi}{3}} 16\,(1 + \cos(2\theta)) - 4 - 8\cos\theta \; d\theta \\
&= \int_0^{\frac{\pi}{3}} 16 + 16\cos(2\theta) - 4 - 8\cos\theta \; d\theta
\end{aligned}
$$

$$= \int_0^{\frac{\pi}{3}} 12 + 16\cos(2\theta) - 8\cos\theta \; d\theta$$

$$= 12\theta + 8\sin(2\theta) - 8\sin\theta \Big|_0^{\frac{\pi}{3}}$$

$$= \left(12\left(\frac{\pi}{3}\right) + 8\sin(\frac{2\pi}{3}) - 8\sin\left(\frac{\pi}{3}\right)\right) - (0 + 0 - 0)$$

$$= 4\pi + 8\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{\sqrt{3}}{2}\right) = \boxed{4\pi}$$

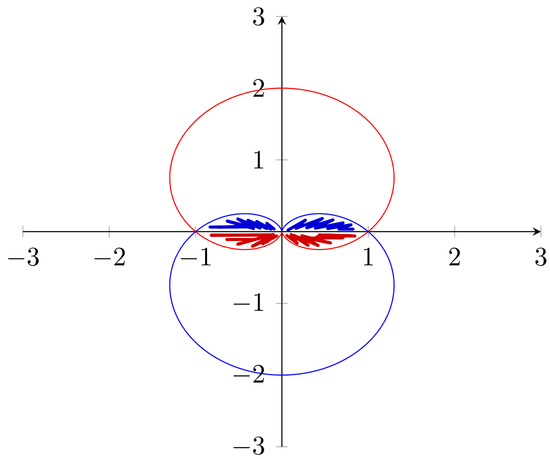
Alternate Option:

Area$=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left((6\cos\theta)^2 - (2 + 2\cos\theta)^2\right) \; d\theta$

Alternate Option:

Area$=\cancel{2}\left(\frac{1}{\cancel{2}}\right)\int_{-\frac{\pi}{3}}^{0} \left((6\cos\theta)^2 - (2 + 2\cos\theta)^2\right) \; d\theta$

3. Compute the area bounded between the polar curves $r = 1 + \sin\theta$ and $r = 1 - \sin\theta$.



These two curves intersect when $1 + \sin\theta = 1 - \sin\theta$ or when $\sin\theta = 0$ so when $\theta = 0, \pi, 2\pi, \ldots$

$$\text{Area} = A = 4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \left((\text{outer } \; r)^2 - (\text{inner } \; r)^2\right) \; d\theta\right)$$

$$= 2\int_0^{\frac{\pi}{2}} (1-\sin\theta)^2 - 0^2 \; d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1 - 2\sin\theta + \sin^2\theta \; d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1 - 2\sin\theta + \left(\frac{1-\cos(2\theta)}{2}\right) \; d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1 - 2\sin\theta + \frac{1}{2} - \frac{\cos(2\theta)}{2} \; d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{3}{2} - 2\sin\theta - \frac{\cos(2\theta)}{2} \; d\theta$$

$$= 2\left(\frac{3}{2}\theta + 2\cos\theta - \frac{\sin(2\theta)}{4}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= 2\left[\left(\frac{3}{2}\left(\frac{\pi}{2}\right) + \cancelto{0}{2\cos\left(\frac{\pi}{2}\right)} - \cancelto{0}{\frac{\sin\pi}{4}}\right) - \left(0 + \cancelto{1}{2\cos 0} - \cancelto{0}{\frac{\sin 0}{4}}\right)\right]$$

$$= 2\left(\frac{3\pi}{4} - 2\right) = \boxed{\frac{3\pi}{2} - 4}$$

Alternate option:

$$\text{Area}{=}A = 4\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1-\sin\theta)^2 \; d\theta\right)$$

Alternate option:

$$\text{Area}{=}A = 4\left(\frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} (1+\sin\theta)^2 \; d\theta\right)$$

Alternate option:

$$\text{Area}{=}A = 4\left(\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi} (1+\sin\theta)^2 \; d\theta\right) \overset{\text{or}}{=} 4\left(\frac{1}{2}\int_{-\frac{\pi}{2}}^{0} (1+\sin\theta)^2 \; d\theta\right)$$

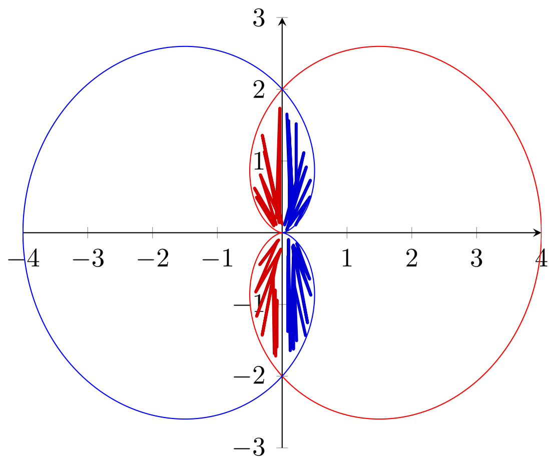
Alternate option:

Area=$A = 2\left(\frac{1}{2}\int_0^{\pi} (1-\sin\theta)^2 \ d\theta\right)$

Alternate option:

Area=$A = 2\left(\frac{1}{2}\int_{\pi}^{2\pi} (1+\sin\theta)^2 \ d\theta\right)$

4. Compute the area bounded between the polar curves $r = 2 + 2\cos\theta$ and $r = 2 - 2\cos\theta$.



These two curves intersect when $2 + 2\cos\theta = 2 - 2\cos\theta$ or when $\cos\theta = 0$ so when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

First we will compute the area by multiplying the area bounded in only the first (cartesian) quadrant by four.

$$\begin{aligned}
\text{Area} = A \quad &= 4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \left((\text{outer} \ \ r)^2 - (\text{inner} \ \ r)^2\right) \ d\theta\right) \\
&= 2\int_0^{\frac{\pi}{2}} (2-2\cos\theta)^2 - 0^2 \ d\theta \\
&= 2\int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + 4\cos^2\theta \ d\theta \\
&= 2\int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + 4\left(\frac{1+\cos(2\theta)}{2}\right) \ d\theta
\end{aligned}$$

$$= 2\int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + 2\left(1 + \cos(2\theta)\right)\ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 4 - 8\cos\theta + 2 + 2\cos(2\theta)\ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 6 - 8\cos\theta + 2\cos(2\theta)\ d\theta$$

$$= 2\left(6\theta - 8\sin\theta + \sin(2\theta)\right)\Big|_0^{\frac{\pi}{2}}$$

$$= 2\left[\left(6\left(\frac{\pi}{2}\right) - 8\cancelto{1}{\sin\frac{\pi}{2}} + \cancelto{0}{\sin\pi}\right) - \left(0 - 8\cancelto{0}{\sin 0} + \cancelto{0}{\sin 0}\right)\right]$$

$$= 2(3\pi - 8) = \boxed{6\pi - 16}$$

Alternate option:

$$\text{Area}=A = 4\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (2 + 2\cos\theta)^2\ \ d\theta\right)$$

Alternate option:

$$\text{Area}=A = 4\left(\frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} (2 + 2\cos\theta)^2\ \ d\theta\right)$$

Alternate option:

$$\text{Area}=A = 4\left(\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi} (2 - 2\cos\theta)^2\ \ d\theta\right) \overset{\text{or}}{=} 4\left(\frac{1}{2}\int_{-\frac{\pi}{2}}^{0} (2 - 2\cos\theta)^2\ \ d\theta\right)$$

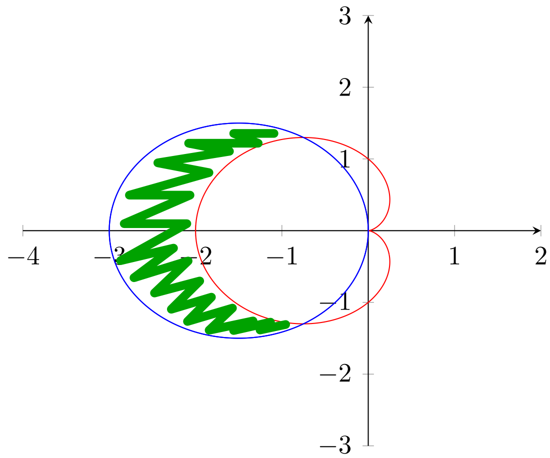
Alternate option:

$$\text{Area}=A = 2\left(\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2 - 2\cos\theta)^2\ \ d\theta\right)$$

Alternate option:

$$\text{Area}=A = 2\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (2 + 2\cos\theta)^2\ \ d\theta\right)$$

5. Compute the area bounded ourside the polar curve $r = 1 - \cos\theta$ and inside $r = -3\cos\theta$.



These two polar curves intersect when $1 - \cos\theta = -3\cos\theta \Rightarrow 2\cos\theta = -1 \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$. Using symmetry, we will integrate from $\theta = \dfrac{2\pi}{3}$ to $\theta = \pi$ and double that area.

$$\begin{aligned}
\text{Area} = A \quad &= \not{2}\left(\frac{1}{\not{2}}\int_{\frac{2\pi}{3}}^{\pi}\left((\text{outer} \;\; r)^2 - (\text{inner} \;\; r)^2\right)\, d\theta\right) \\
&= \int_{\frac{2\pi}{3}}^{\pi}\left((-3\cos\theta)^2 - (1-\cos\theta)^2\right)\, d\theta \\
&= \int_{\frac{2\pi}{3}}^{\pi} 9\cos^2\theta - (1 - 2\cos\theta + \cos^2\theta)\; d\theta \\
&= \int_{\frac{2\pi}{3}}^{\pi} 9\cos^2\theta - 1 + 2\cos\theta - \cos^2\theta\; d\theta \\
&= \int_{\frac{2\pi}{3}}^{\pi} 8\cos^2\theta - 1 + 2\cos\theta\; d\theta \\
&= \int_{\frac{2\pi}{3}}^{\pi} 8\left(\frac{1+\cos(2\theta)}{2}\right) - 1 + 2\cos\theta\; d\theta \\
&= \int_{\frac{2\pi}{3}}^{\pi} 4\left(1+\cos(2\theta)\right) - 1 + 2\cos\theta\; d\theta
\end{aligned}$$

$$= \int_{\frac{2\pi}{3}}^{\pi} 4 + 4\cos(2\theta) - 1 + 2\cos\theta \; d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\pi} 3 + 4\cos(2\theta) + 2\cos\theta \; d\theta$$

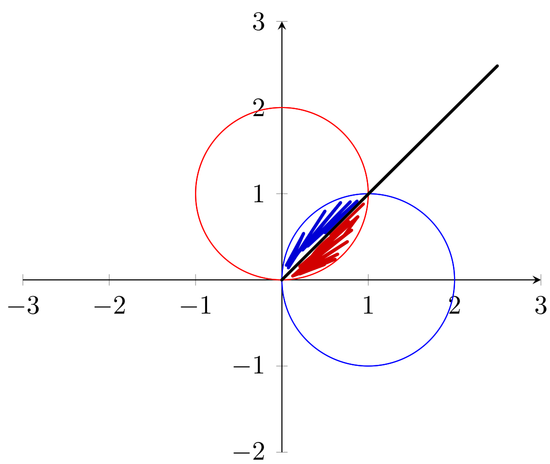
$$= 3\theta + 2\sin(2\theta) + 2\sin\theta \Big|_{\frac{2\pi}{3}}^{\pi}$$

$$= \left(3\pi + 2\sin(2\pi)^{0} + 2\sin\pi^{0}\right) - \left(3\left(\frac{2\pi}{3}\right) + 2\sin\left(\frac{4\pi}{3}\right)^{-\frac{\sqrt{3}}{2}} + 2\sin\left(\frac{2\pi}{3}\right)^{\frac{\sqrt{3}}{2}}\right)$$

$$= 3\pi - 2\pi + 2\left(\frac{+\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right) = \boxed{\pi}$$

Note: This is the same area bounded in problem 1 above, since this is a rotated version of that bounded region.

6. Compute the area **inside both** of the polar curves $r = 2\cos\theta$ and $r = 2\sin\theta$.



These two polar circles intersect when $2\cos\theta = 2\sin\theta$, so when $\theta = \frac{\pi}{4}$. Using symmetry, we will integrate from $\theta = 0$ to $\theta = \frac{\pi}{4}$ and double that area.

$$\text{Area} = A = \cancel{2}\left(\frac{\cancel{1}}{\cancel{2}}\int_0^{\frac{\pi}{4}} \left((\text{outer} \;\; r)^2 - (\text{inner} \;\; r)^2\right) \; d\theta\right)$$

$$= \int_0^{\frac{\pi}{4}} (2\sin\theta)^2 - 0 \; d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sin^2\theta \; d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \frac{1-\cos(2\theta)}{2} \; d\theta$$

$$= \frac{4}{2}\int_0^{\frac{\pi}{4}} 1-\cos(2\theta) \; d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} 1-\cos(2\theta) \; d\theta$$

$$= 2\left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_0^{\frac{\pi}{4}}$$

$$= 2\left(\frac{\pi}{4} - \frac{\sin\left(2\left(\frac{\pi}{4}\right)\right)}{2} - \left(0 - \cancelto{0}{\sin 0}\right)\right)$$

$$= 2\left(\frac{\pi}{4} - \frac{\cancelto{1}{\sin\left(\frac{\pi}{2}\right)}}{2}\right) = 2\left(\frac{\pi}{4}\right) - \cancel{2}\left(\frac{1}{\cancel{2}}\right) = \boxed{\frac{\pi}{2} - 1}$$

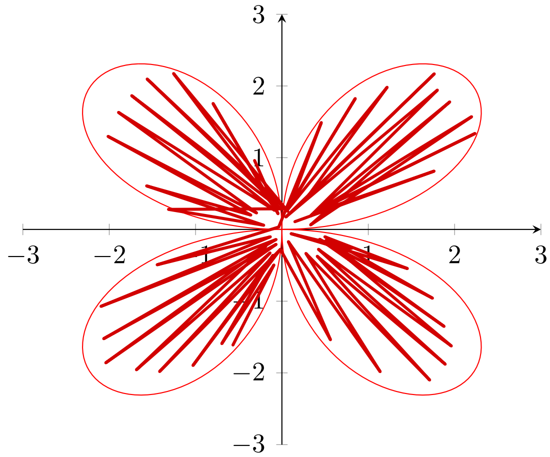
Alternate option:

$$\text{Area} = A = \cancel{2}\left(\frac{\cancel{1}}{\cancel{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos\theta)^2 \; d\theta\right)$$

Alternate option:

$$\text{Area} = A = \frac{1}{2}\int_0^{\frac{\pi}{4}} (2\sin\theta)^2 \; d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\cos\theta)^2 \; d\theta$$

7. Compute the area bounded in all 4 petal loops of the polar curve $r = 3\sin(2\theta)$.



$$\text{Area} = A = 4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \left((\text{outer } r)^2 - (\text{inner } r)^2\right)\, d\theta\right)$$

$$= 2\int_0^{\frac{\pi}{2}} (3\sin(2\theta))^2\, d\theta$$

$$= 2\cdot 9\int_0^{\frac{\pi}{2}} \sin^2(2\theta)\, d\theta$$

$$= 18\int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2}\, d\theta$$

$$= \frac{18}{2}\int_0^{\frac{\pi}{2}} 1-\cos(4\theta)\, d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} 1-\cos(4\theta)\, d\theta = 9\left(\theta - \frac{\sin(4\theta)}{4}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= 9\left[\left(\frac{\pi}{2} - \frac{\cancelto{0}{\sin(2\pi)}}{4}\right) - \left(0 - \frac{\cancelto{0}{\sin 0}}{4}\right)\right] = \boxed{\frac{9\pi}{2}}$$

Alternate Option:

$$\text{Area=}A = \frac{1}{2}\int_0^{2\pi} (3\sin(2\theta))^2\, d\theta$$

Alternate Option:

$$\text{Area=}A = 2\left(\frac{1}{2}\int_0^{\pi} (3\sin(2\theta))^2\, d\theta\right)$$

Alternate Option:

$$\text{Area}=A = 2\left(\frac{1}{2}\int_{\pi}^{2\pi} (3\sin(2\theta))^2 \;\; d\theta\right)$$

Alternate Option:

$$\text{Area}=A = 4\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (3\sin(2\theta))^2 \;\; d\theta\right)$$

Alternate Option:

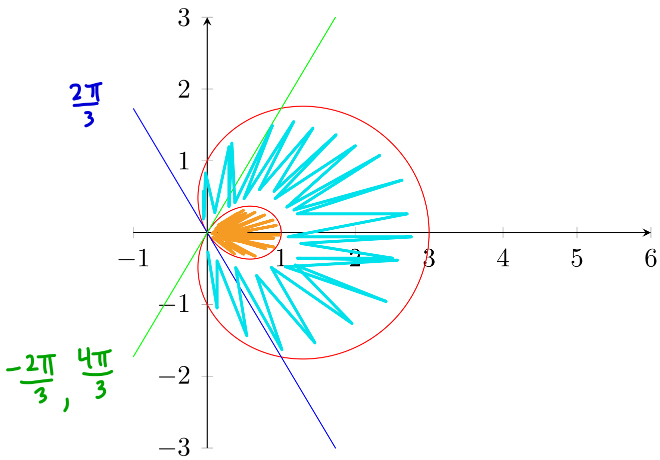
$$\text{Area}=A = 4\left(\frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}} (3\sin(2\theta))^2 \;\; d\theta\right)$$

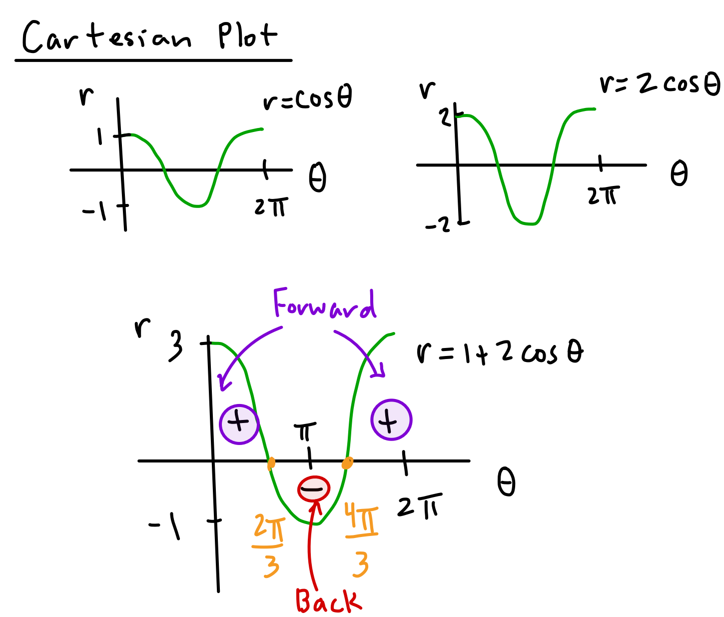
Alternate Option:

$$\text{Area}=A = 4\left(\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi} (3\sin(2\theta))^2 \;\; d\theta\right)$$

Challenge:

8. (a) Sketch the polar curve $r = 1 + 2\cos\theta$. Use the Cartesian plot to discover the Polar plot.





Set the polar curve equal to 0.

$r = 1 + 2\cos\theta = 0 \Rightarrow 2\cos\theta = -1 \Rightarrow \cos\theta = -\dfrac{1}{2}$ which yields $\theta = \dfrac{2\pi}{3}$ or $\theta = \dfrac{4\pi}{3}$

(b) Set-up, **BUT DO NOT EVALUATE!!**, the definite integral representing the area inside the larger loop.

We can integrate from $\theta = -\frac{2\pi}{3}$ to $\theta = \frac{2\pi}{3}$ for the larger loop, OR we can integrate from $\theta = 0$ to $\theta = \frac{2\pi}{3}$ and double using symmetry.

$$\text{Area}=A = 2\left(\frac{1}{2}\int_{0}^{\frac{2\pi}{3}} (1+2\cos\theta)^2 \;\, d\theta\right)$$

or

$$\text{Area}=A = 2\left(\frac{1}{2}\int_{-\frac{2\pi}{3}}^{0} (1+2\cos\theta)^2 \;\, d\theta\right)$$

or

$$\text{Area}=A = \frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} (1+2\cos\theta)^2 \;\, d\theta$$

(c) Set-up, **BUT DO NOT EVALUATE!!**, the definite integral representing the area inside the smaller loop.

We can integrate from $\theta = \frac{2\pi}{3}$ to $\theta = \frac{4\pi}{3}$ for the inner loop, OR we can integrate from $\theta = \frac{2\pi}{3}$ to $\theta = \pi$ and double using symmetry.

$$\text{Area}=A = 2\left(\frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (1+2\cos\theta)^2 \;\, d\theta\right)$$

or

$$\text{Area}=A = 2\left(\frac{1}{2}\int_{\pi} \frac{4\pi}{3} (1+2\cos\theta)^2 \;\, d\theta\right)$$

or

$$\text{Area}=A = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1+2\cos\theta)^2 \;\, d\theta$$