Extra Examples of Trigonometric Substitutions

Math 121 D. Benedetto

1.

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx$$

Recognize **difference** of squares under the square root

$$= \int \frac{(2\sin\theta)^2}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\;d\theta$$

Use a sine sub for $a^2 - x^2$. **Remember,** sub for dx

$$= \int \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}}\,2\cos\theta\;d\theta$$

Work the algebra to create the identity $1 - \sin^2\theta = \cos^2\theta$

$$= 4\int \frac{\sin^2\theta}{\sqrt{4}\sqrt{\cos^2\theta}}\,2\cos\theta\;d\theta$$

The identity creates the perfect square under the root

$$= 4\int \frac{\sin^2\theta}{2\cos\theta}\,2\cos\theta\;d\theta$$

Simplify the root of the perfect square, see what cancels

$$= 4\int \sin^2\theta\;d\theta$$

Prepare for the **EVEN power technique**

$$= 4\int \frac{1-\cos(2\theta)}{2}\;d\theta$$

Using the half angle identity

$$= 2\int 1-\cos(2\theta)\;d\theta$$

Using a u-sub if needed

$$= 2\left(\theta - \frac{\sin(2\theta)}{2}\right) + C$$

$$= 2\left(\theta - \frac{2\sin\theta\cos\theta}{2}\right) + C$$

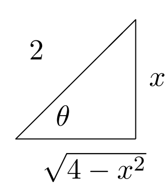
Using the Double Angle Identity

$$= 2\left(\theta - \sin\theta\cos\theta\right) + C$$

Using the Trig Triangle to *unwind* the Trig Sub

$$= 2\left(\arcsin\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)\right) + C \quad = \boxed{2\arcsin\left(\frac{x}{2}\right) - \left(\frac{x\sqrt{4-x^2}}{2}\right) + C}$$

Trig. Substitute $\boxed{\begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta d\theta \end{array}}$



2. Try the same problem in a definite integral. You can either change your limits with the trig sub **or** mark them as x limits. You should choose only one method below.

Option 1: Changing Limits

$$\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(2\sin\theta)^2}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\;d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}}\,2\cos\theta\;d\theta$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2\theta}{\sqrt{4}\sqrt{\cos^2\theta}}\,2\cos\theta\;d\theta = 4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2\theta}{2\cos\theta}\,2\cos\theta\;d\theta = 4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta\;d\theta$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos(2\theta)}{2}\,d\theta = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1-\cos(2\theta)\;d\theta = 2\left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= 2\left(\left(\frac{\pi}{3} - \frac{\sin\left(\frac{2\pi}{3}\right)}{2}\right) - \left(\frac{\pi}{6} - \frac{\sin\left(\frac{\pi}{3}\right)}{2}\right)\right) = 2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) = \boxed{\frac{\pi}{3}}$$

Recall: $x = 2\sin\theta \Rightarrow \theta = \arcsin\left(\frac{x}{2}\right)$.

Change Limits:

$$\boxed{\begin{array}{l} x = 1 \Rightarrow \theta = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \\[2ex] x = \sqrt{3} \Rightarrow \theta = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \end{array}}$$

Note: If you change your limits, you do not need the Trig Triangle to return to x-variable.

Option 2: Mark Limits

$$\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_{x=1}^{x=\sqrt{3}} \frac{(2\sin\theta)^2}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\;d\theta = \cdots = 2\left(\theta - \sin\theta\cos\theta\right)\Big|_{x=1}^{x=\sqrt{3}}$$

using antiderivative from above in 1.

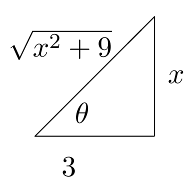
$$= 2\arcsin\left(\frac{x}{2}\right) - \left(\frac{x\sqrt{4-x^2}}{2}\right)\Bigg|_1^{\sqrt{3}} = 2\arcsin\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}\sqrt{4-3}}{2}\right) - \left[2\arcsin\left(\frac{1}{2}\right) - \left(\frac{1\sqrt{4-1}}{2}\right)\right]$$

$$2\left(\frac{\pi}{3}\right) - \frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} = 2\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 2\left(\frac{\pi}{6}\right) = \boxed{\frac{\pi}{3}}$$

3.

$\int \frac{1}{(9+x^2)^{\frac{5}{2}}}\ dx$	Recognize **sum** of squares under the square root
$= \int \frac{1}{\left(\sqrt{9+9\tan^2\theta}\right)^5}\ 3\sec^2\theta\ d\theta$	Use a tangent sub for a^2+x^2. **Remember**, sub for dx
$= \int \frac{1}{\left(\sqrt{9(1+\tan^2\theta)}\right)^5}\ 3\sec^2\theta\ d\theta$	Work the algebra to create the identity $1+\tan^2\theta=\sec^2\theta$
$= \int \frac{1}{\left(\sqrt{9}\sqrt{\sec^2\theta}\right)^5}\ 3\sec^2\theta\ d\theta$	The identity creates the perfect square under the root
$= \frac{3}{3^5}\int \frac{1}{\sec^5\theta}\ \sec^2\theta\ d\theta$	Simplify the root of the perfect square, see what cancels
$= \frac{1}{3^4}\int \frac{1}{\sec^3\theta}\ d\theta$	Flip the secant to cosine
$= \frac{1}{3^4}\int \cos^3\theta\ d\theta$	**Prepare for the ODD power technique**
$= \frac{1}{81}\int \cos^2\theta\ \cos\theta\ d\theta$	*Isolate* one copy of cosine
$= \frac{1}{81}\int \left(1-\sin^2\theta\right)\ cos\theta\ d\theta$	*Convert* using trig identity
$= \frac{1}{81}\int 1-w^2\ dw$	*Finish* with a substitution
$= \frac{1}{81}\left(w-\frac{w^3}{3}\right)+C$	
$= \frac{1}{81}\left(\sin\theta-\frac{\sin^3\theta}{3}\right)+C$	$= \boxed{\frac{1}{81}\left(\frac{x}{\sqrt{9+x^2}}-\frac{1}{3}\left(\frac{x}{\sqrt{9+x^2}}\right)^3\right)+C}$

Trig. Substitute $\boxed{\begin{array}{l} u = 3\tan\theta \\ du = 3\sec^2\theta d\theta \end{array}}$



Standard substitution $\boxed{\begin{array}{l} w = \sin\theta \\ dw = \cos\theta\ d\theta \end{array}}$

$$4.\ \int x\arcsin x\ dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\ dx$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\ d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cdot\cos\theta\ d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta\ d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2}d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1-\cos(2\theta)d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}(2\sin\theta\cos\theta) + C = \boxed{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$$

IBP to start:

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = xdx \\ du = \frac{1}{\sqrt{1-x^2}}dx & v = \frac{x^2}{2} \end{array}}$$

Trig. Substitute

$$\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta d\theta \end{array}}$$