Take Home Quiz 6 Fall 2023 Answers

1. $$\sum_{n=1}^{\infty} \frac{(-1)^n (3x+1)^n}{(n+7)\cdot 7^n}$$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}\,(3x+1)^{n+1}}{(n+8)\,7^{n+1}}}{\frac{\cancel{(-1)^{n}}\,(3x+1)^{n}}{(n+7)\,7^{n}}}\right| = \lim_{n\to\infty}\overset{|3x+1|}{\left|\frac{(3x+1)^{n+1}}{(3x+1)^n}\right|}\cdot\underbrace{\frac{\cancel{n+7}}{n+8}}_{\to 1}\cdot\frac{\cancel{7^n}}{\underset{\cancel{7^n}\cdot\boxed{7}}{7^{n+1}}}$$

$$= \lim_{n\to\infty}\frac{|3x+1|}{7} = \frac{|3x+1|}{7} < 1$$

Converges by Ratio Test when

$|3x+1| < 7$ $\quad -7 < 3x+1 < 7$

$\quad\quad -1 \quad\;\; -1 \quad\; -1$

$-8 < 3x < 6$

$-8/3 < x < 2$

Manually Test Convergence at Endpoints

Take $x = -8/3$. Series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n\left[\cancel{3}\left(-\frac{8}{\cancel{3}}\right)+1\right]^n}{(n+7)\cdot 7^n} = \sum_{n=1}^{\infty}\frac{\overset{(*)\ (-1)^n\cancel{7^n}}{(-1)^n(-7)^n}}{(n+7)\ \cancel{7^n}} = \sum_{n=1}^{\infty}\frac{\overset{\text{Even}}{\cancel{(-1)^{2n}}}\to 1}{n+7} = \sum_{n=1}^{\infty}\frac{1}{n+7} \approx \sum_{n=1}^{\infty}\frac{1}{n}$$

Diverges p-Series $p = 1$.

(*) OR $(-1)^n(-7)^n = [(-1)(-7)]^n = 7^n$

OR $(-1)^n(-1)^n 7^n = [(-1)(-1)]^n 7^n = \cancel{1^n}\,\overset{1}{}7^n$

$$\lim_{n\to\infty}\frac{\frac{1}{n+7}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\cancel{n}}{\cancel{n+7}}\to 1 = 1$$

Finite, Non-Zero

$\Rightarrow$ Series Diverges by LCT

Take $x = 2$. Series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n\overset{7^n}{\cancel{(3(2)+1)^n}}}{(n+7)\ \cancel{7^n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+7}$$

Converges by AST

① $b_n = \dfrac{1}{n+7} > 0$

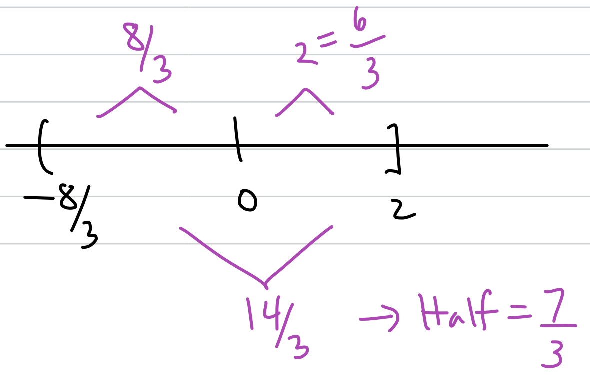
② $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\underset{\infty}{\cancel{n+7}}}\to 0 = 0$

③ Terms decreasing

$b_{n+1} = \dfrac{1}{n+8} \le \dfrac{1}{n+7} = b_n$

Finally, $I = \left(-\frac{8}{3}, 2\right]$

$R = \dfrac{7}{3}$



2. $$\sum_{n=1}^{\infty} n^n\cdot(\ln n)(x-6)^n$$

Ratio Test

$$(n+1)^n\cdot(n+1)$$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}\cdot(\ln(n+1))(x-6)^{n+1}}{n^n\cdot(\ln n)\cdot(x-6)^n}\right| = \lim_{n\to\infty}\cancelto{e}{\frac{(n+1)^n}{n^n}}\cdot(n+1)\,\cancelto{1}{\frac{\ln(n+1)}{\ln n}}\cdot|x-6|$$

See ☆

$$= \lim_{n\to\infty} e\,\cancelto{\infty}{(n+1)}\cdot(1)\cdot|x-6| = \infty > 1$$

Diverges by Ratio Test

for all x unless x=6 ↑

when $L=0<1$

☆ $$\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{x}{x+1} \underset{L'H}{=} \lim_{x\to\infty}\frac{1}{1} = 1$$

Finally, $$\boxed{\begin{array}{l} I=\{6\} \\ R=0 \end{array}}$$

3. $$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

Note: yes, this is Cosine with $R=\infty$ but we want to justify

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}\,x^{2(n+1)}}{(2(n+1))!}}{\frac{\cancel{(-1)^n}\,x^{2n}}{(2n)!}}\right| = \lim_{n\to\infty}\overset{|x|^2}{\left|\frac{x^{2n+2}}{x^{2n}}\right|}\cdot\frac{\cancel{(2n)!}}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)\cancel{(2n)!}}} = \lim_{n\to\infty}\cancelto{0}{\frac{|x|^2}{\underset{\infty}{(2n+2)(2n+1)}}} = 0<1$$

Always 0 for all x

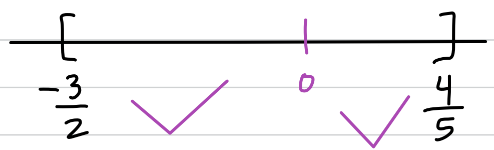
Converges by Ratio Test

for all x in ℝ

Finally, $$\boxed{\begin{array}{l} I=(-\infty,\infty) \\ R=\infty \end{array}}$$

Optional Bonus: Need Series with Interval of Convergence $I=\left[-\frac{3}{2},\frac{4}{5}\right]$

First picture the Interval on the Number Line to find Radius of Convergence

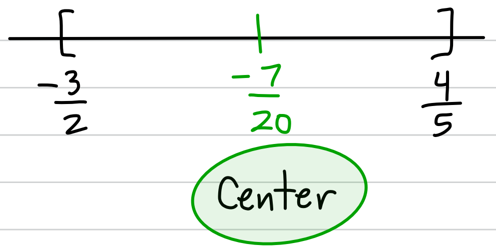


$$\frac{3}{2} = \frac{15}{10} \qquad \frac{4}{5} = \frac{8}{10}$$

$$\frac{23}{10}$$ ← Total Length ⇒ Radius of Convergence is Half $R=\frac{23}{20}$

Next, find the Center Point using one endpoint and the Radius above

center $a = \frac{4}{5} - \frac{23}{20} = \frac{16}{20} - \frac{23}{20} = -\frac{7}{20}$



Therefore, for any x to be contained in I, we need $\left|x-\left(-\frac{7}{20}\right)\right| < \frac{23}{20}$

That is, $\left|x+\frac{7}{20}\right| < \frac{23}{20}$ OR $\frac{20}{23}\left|x+\frac{7}{20}\right| < 1$ OR $\frac{|20x+7|}{23} < 1$

Looks like end of a Ratio Test Computation

Guess. $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n (20x+7)^n}{n^2 (23)^n}}$ Run Ratio Test

OR Some power piece with p>2 to yield convergence at both endpoints

one copy $|20x+7|$

$\left(\frac{\cancel{n}}{\cancel{n+1}}\right)^2 \to 1$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(20x+7)^{n+1}}{(n+1)^2(23)^{n+1}}}{\frac{\cancel{(-1)^n}(20x+7)^n}{n^2(23)^n}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(20x+7)^{n+1}}}{\cancel{(20x+7)^n}}\right|\cdot\frac{n^2}{(n+1)^2}\cdot\frac{\cancel{(23)^n}}{\cancel{(23)^{n+1}}} = \frac{|20x+7|}{23} < 1$$

$\frac{1}{23}$

Converges by Ratio Test when

OR when $|20x+7| < 23 \Rightarrow -23 < 20x+7 < 23 \Rightarrow -\frac{3}{2} < x < \frac{4}{5}$ good!

$-7 \quad -7 \quad -7$

$$\frac{-30 < 20x < 16}{20}$$

Manually check Convergence at Endpoints.

Test $x = \frac{-3}{2}$. Series becomes

$(-1)^n(23)^n$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\left[\overset{10}{\cancel{20}}\left(\frac{-3}{\cancel{2}}\right)+7\right]^n}{n^2\,(23)^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-23)^n}{n^2(23)^n} = \sum_{n=1}^{\infty}\frac{\cancel{(-1)^{\overset{\text{even}}{2n}}}\cdot\cancel{(23)^n}}{n^2\,\cancel{(23)^n}} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$\to 1$

Convergent p-Series $p = 2 > 1$

Take $x = \frac{4}{5}$. Series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n\left(\overset{4}{\cancel{20}}\left(\frac{4}{\cancel{5}}\right)+7\right)^n}{n^2\cdot(23)^n} = \sum_{n=1}^{\infty}\frac{(-1)^n\cancel{(23)^n}}{n^2\cdot\cancel{(23)^n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \hookrightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}$

A.S. Converges p-Series $p=2>1$

ACT $\Leftarrow$

O.S Converges by ACT

Test using AST [✓ OR] ACT

OR AST works...

So both endpoints included in the Domain; therefore

$I = \left[-\frac{3}{2}, \frac{4}{5}\right]$ as desired