Take Home Quiz 5 Fall 2023 Answers

1. $\sum_{n=1}^{\infty} n^7+4$ Diverges by nTDT because $\lim_{n\to\infty} n^7+4 = \infty \neq 0$

2. $\sum_{n=1}^{\infty} \frac{n^3}{n^4+7} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Divergent p-Series $p=1$ (OR, Harmonic)

(note: true bound BUT not in helpful order

$\frac{n^3}{n^4+7} \le \frac{n^3}{n^4} = \frac{1}{n}$ "Smaller than Diverge is Inconclusive")

need LCT Limit

$$\lim_{n\to\infty} \frac{\frac{n^3}{n^4+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^4}{n^4+7} \cdot \frac{1/n^4}{1/n^4} = \lim_{n\to\infty} \frac{1}{1+\frac{7}{n^4}\xrightarrow{}0} = 1$$

(Finite Non-Zero

"Share Same Behavior"
"Comparable"

$\Rightarrow$ Original Series also Diverges by LCT

3. $\sum \frac{n^4+7}{4n^7+1} \approx \sum \frac{n^4}{n^7} = \sum \frac{1}{n^3}$ Converges p-Series $p=3>1$

$$\lim_{n\to\infty} \frac{\frac{n^4+7}{4n^7+1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^7+7n^3}{4n^7+1} \cdot \frac{\frac{1}{n^7}}{\frac{1}{n^7}} = \lim_{n\to\infty} \frac{1+\frac{7}{n^4}\to 0}{4+\frac{1}{n^7}\to 0} \to \frac{1}{4} = \frac{1}{4}$$

Finite and Non-Zero

"Comparable"

$\Rightarrow$ Series also Converges by LCT

4. $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n \cdot n!}{n^n \cdot n^6}$ Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}4^{n+1}(n+1)!}{(n+1)^{n+1}(n+1)^6}}{\frac{(-1)^n 4^n n!}{n^n \cdot n^6}} \right| \text{ Flip Partner} = \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \cdot \frac{n^6}{(n+1)^6}$$

$4^n \cdot 4$, $(n+1)\,n!$, $(n+1)^n \cdot (n+1)$, $\left(\frac{n \cdot \frac{1}{n}}{n+1 \cdot \frac{1}{n}}\right)^6$

$$= \lim_{n\to\infty} 4 \cdot \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} \cdot \left(\frac{1}{1+\frac{1}{n}}\right)^6 \to 1 = \frac{4}{e} > 1$$

Diverges by Ratio Test

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5}$

First: Comparison Test

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n+5} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$$ Converges by p-Series $p=2>1$

Bound Terms

$$\frac{1}{n^2+2n+5} \leq \frac{1}{n^2}$$ $\Rightarrow$ Original Series also Converges by the Comparison Test

Second: Integral Test

Study the Improper Integral

Complete Square

$$\int_1^{\infty} \frac{1}{x^2+2x+5}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+2x+5}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{(x+1)^2+4}\,dx$$

Discriminant:

$b^2-4ac = 4-4(1)(5) = -16 < 0$

$\boxed{\begin{array}{l} u = x+1 \\ du = dx \end{array}}$ $\boxed{\begin{array}{l} x=1 \Rightarrow u=2 \\ x=t \Rightarrow u=t+1 \end{array}}$

$$= \lim_{t\to\infty} \int_2^{t+1} \frac{1}{u^2+4}\,du$$

$$= \lim_{t\to\infty} \frac{1}{2} \arctan\left(\frac{u}{2}\right)\Big|_2^{t+1}$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\arctan\left(\frac{t+1}{2}\right) - \arctan\left(\frac{2}{2}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$ Integral Converges

$\Rightarrow$ Original Series also Converges by the Integral Test