Take Home Quiz 3 Answers Fall 2023

1. IBP

$$\int_0^1 (x+1)\arctan x\,dx = \left(\frac{x^2}{2}+x\right)\arctan x\Big|_0^1 - \int_0^1 \frac{\frac{x^2}{2}+x}{1+x^2}\,dx$$

distribute; split–split

$u=\arctan x$	$dv = x+1\,dx$
$du = \frac{1}{1+x^2}$	$v = \frac{x^2}{2}+x$

$$= \left(\frac{x^2}{2}+x\right)\arctan x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{1+x^2}\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$$

slip-in/slip-out; u-sub.

$$= \left(\frac{x^2}{2}+x\right)\arctan x\Big|_0^1 - \frac{1}{2}\left[\int_0^1 \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\,dx\right] - \frac{1}{2}\int_1^2 \frac{1}{u}\,du$$

$u = 1+x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$= \left(\frac{x^2}{2}+x\right)\arctan x\Big|_0^1 - \frac{1}{2}\Big[x - \arctan x\Big]\Big|_0^1 - \frac{1}{2}\ln|u|\Big|_1^2$$

$x=0 \Rightarrow u=1$
$x=1 \Rightarrow u=2$

$$= \frac{3}{2}\arctan 1 - 0 - \frac{1}{2}\Big[1-\arctan 1 - (0-0)\Big] - \frac{1}{2}(\ln 2 - \ln 1)$$

($\arctan 1 = \pi/4$, $\ln 1 = 0$)

$$= \frac{3\pi}{8} - \frac{1}{2} + \frac{\pi}{8} - \frac{\ln 2}{2}$$

$$= \frac{4\pi}{8} - \frac{1}{2} - \frac{\ln 2}{2} = \boxed{\frac{\pi - 1 - \ln 2}{2}}$$

($\frac{4\pi}{8} = \pi/2$) Match!

OR // Alternate Solution ↳ Split Original Integral

1. $$\int_0^1 (x+1)\arctan x\,dx = \int_0^1 x\arctan x\,dx + \int_0^1 \arctan x\,dx$$

HW Question ① ; Done in Clas ②

$u=\arctan x$	$dv = x\,dx$
$du = \frac{1}{1+x^2}dx$	$v = \frac{x^2}{2}$

① $$\int_0^1 x\arctan x\,dx = \frac{x^2}{2}\arctan x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx$$

$$= \frac{x^2}{2}\arctan x\Big|_0^1 - \frac{1}{2}\left(\int_0^1 \frac{x^2+1}{x^2+1}\,dx - \int_0^1 \frac{1}{x^2+1}\,dx\right)$$

$$= \frac{x^2}{2}\arctan x\Big|_0^1 - \frac{1}{2}\big(x - \arctan x\big)\Big|_0^1$$

$$= \frac{x^2}{2}\arctan x\Big|_0^1 - \frac{1}{2}x + \frac{1}{2}\arctan x\Big|_0^1$$

$$= \frac{1}{2}\arctan 1 - 0 - \frac{1}{2} + \frac{1}{2}\arctan 1 - (-0+0)$$

($\arctan 1 = \pi/4$)

$$= \pi/8 - \frac{1}{2} + \frac{\pi}{8} = \boxed{\frac{\pi}{4} - \frac{1}{2}}$$ piece ①

1. Continued

$u = \arctan x \quad dv = dx$

$du = \frac{1}{1+x^2}\,dx \quad v = x$

② $\int_0^1 \arctan x\,dx = x\arctan x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$

: u-sub

$= x\arctan x\Big|_0^1 - \frac{1}{2}\ln|1+x^2|\Big|_0^1$

$= \arctan 1 - 0 - \frac{1}{2}\ln 2 + \frac{1}{2}\ln 1$ ($\arctan 1 = \frac{\pi}{4}$, $\ln 1 = 0$)

$= \frac{\pi}{4} - \frac{\ln 2}{2}$ piece ②

Combine ① + ② $= \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{4} - \frac{\ln 2}{2} = \frac{2\pi}{4} - \frac{1}{2} - \frac{\ln 2}{2} = \frac{\pi - 1 - \ln 2}{2}$

Matches above

2. $\int \frac{1}{(4+x^2)^{7/2}}\,dx = \int \frac{1}{\left(\sqrt{4+x^2}\right)^7}\,dx = \int \frac{1}{\left(\sqrt{4+4\tan^2\theta}\right)^7} \cdot 2\sec^2\theta\,d\theta$

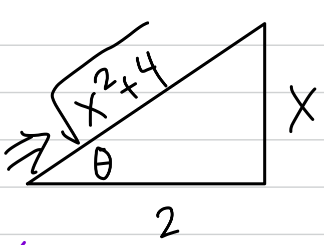
$4(1+\tan^2\theta)$

$4\sec^2\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\tan\theta = \frac{x}{2}$



$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

$= \int \frac{1}{(2\sec\theta)^7} \cdot 2\sec^2\theta\,d\theta = \frac{2}{2^7}\int \frac{\sec^2\theta}{\sec^7\theta}\,d\theta$

$2^7 \cdot \sec^7\theta$

$= \frac{1}{2^6}\int \frac{1}{\sec^5\theta}\,d\theta = \frac{1}{64}\int \cos^5\theta\,d\theta$ ODD Power

$\cos^4\theta \cdot \cos\theta$

$(\cos^2\theta)^2$

$= \frac{1}{64}\int (1-\sin^2\theta)^2 \cdot \cos\theta\,d\theta = \frac{1}{64}\int (1-u^2)^2\,du$

$1 - 2u^2 + u^4$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$= \frac{1}{64}\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right) + C = \frac{1}{64}\left(\sin\theta - \frac{2}{3}\sin^3\theta + \frac{\sin^5\theta}{5}\right) + C$

$= \frac{1}{64}\left(\frac{x}{\sqrt{x^2+4}} - \frac{2}{3}\left(\frac{x}{\sqrt{x^2+4}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{x^2+4}}\right)^5\right) + C$

OR

$= \frac{1}{64}\left(\frac{x}{\sqrt{x^2+4}} - \frac{2x^3}{3(x^2+4)^{3/2}} + \frac{x^5}{5(x^2+4)^{5/2}}\right) + C$

3. $\int \arcsin x \, dx = \int \underset{I}{\overset{u}{\arcsin x}} \cdot \underset{P}{\overset{dv}{1}} \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx$

$w = 1-x^2$
$dw = -2x\,dx$
$-\frac{1}{2} dw = x\,dx$

$u = \arcsin x$	$dv = 1\,dx$
$du = \frac{1}{\sqrt{1-x^2}} dx$	$v = x$

$= x \arcsin x + \frac{1}{2} \int \frac{1}{\sqrt{w}} \, dw$ ($w^{-1/2}$)

$= x \arcsin x + \frac{1}{2} \cdot \frac{w^{1/2}}{\frac{1}{2}} + C$

$= \boxed{x \arcsin x + \sqrt{1-x^2} + C}$