Extra Examples of Improper Integrals

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For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

1. $\displaystyle\int_8^{\infty} \frac{1}{x^2-10x+28}\,dx \lim_{t\to\infty} \int_8^{t} \frac{1}{x^2-10x+28}\,dx$

$= \displaystyle\lim_{t\to\infty} \int_8^{t} \frac{1}{(x-5)^2+3}\,dx$ complete the square

Substitute

$u = x - 5$	$x = 8 \Rightarrow u = 3$
$du = dx$	$x = t \Rightarrow u = t - 5$

$$= \lim_{t\to\infty} \int_3^{t-5} \frac{1}{u^2+3}\,du = \lim_{t\to\infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_3^{t-5} = \lim_{t\to\infty} \frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-5}{\sqrt{3}}\right) - \arctan\left(\frac{3}{\sqrt{3}}\right)\right)$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-5}{\sqrt{3}}\right) - \arctan\left(\sqrt{3}\right)\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{3\pi}{6} - \frac{2\pi}{6}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \boxed{\frac{\pi}{6\sqrt{3}}}$$ Converges

using the formula $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$ and $\displaystyle\lim_{p\to\infty} \arctan p = \frac{\pi}{2}$

2. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{x\ \ln x}\,dx = \lim_{t\to 0^+} \int_t^{\frac{1}{2}} \frac{1}{x\ \ln x}\,dx$

$$= \lim_{t\to 0^+} \int_{\ln t}^{\ln\left(\frac{1}{2}\right)} \frac{1}{u}\,du = \lim_{t\to 0^+} \ln|\ln x|\Bigg|_{\ln t}^{\ln\left(\frac{1}{2}\right)}$$

$$= \lim_{t\to 0^+} \ln\left|\ln\left(\frac{1}{2}\right)\right| - \ln|\ln t| = \ln\left|\ln\left(\frac{1}{2}\right)\right| - \infty = \boxed{-\infty}$$ Diverges

Substitute

$u = \ln x$	$x = t \Rightarrow u = \ln t$
$du = \frac{1}{x}dx$	$x = \frac{1}{2} \Rightarrow u = \ln\left(\frac{1}{2}\right)$

NOTE: $\displaystyle\lim_{t\to 0^+} \ln t = -\infty$ and then $\displaystyle\lim_{p\to\infty} \ln p = \infty$

3. $\displaystyle\int_7^{\infty} \frac{4}{x^2-8x+12}\,dx = \lim_{t\to\infty}\int_7^{t} \frac{4}{x^2-8x+12}\,dx$

$\displaystyle = \lim_{t\to\infty}\int_7^{t} \frac{4}{(x-6)(x-2)}\,dx = \lim_{t\to\infty}\int_7^{t} \frac{1}{x-6} - \frac{1}{x-2}\,dx$ See PFD below

$\displaystyle = \lim_{t\to\infty} \ln|x-6| - \ln|x-2|\Big|_7^t = \lim_{t\to\infty} \ln|t-6| - \ln|t-2| - (\ln|1| - \ln|5|)$

(Indeterminate Difference)

$\displaystyle = \lim_{t\to\infty} \ln\left|\frac{t-6}{t-2}\right| - 0 + \ln 5 = \ln\left|\lim_{t\to\infty}\frac{t-6}{t-2}\right| + \ln 5 \overset{\text{L'H}}{=} \ln\left|\lim_{t\to\infty}\frac{1}{1}\right| + \ln 5 = 0 + \ln 5$

$= \boxed{\ln 5}$ Converges

Partial Fractions Decomposition:

$\displaystyle \frac{4}{(x-6)(x-2)} = \frac{A}{x-6} + \frac{B}{x-2}$

Clearing the denominator yields:

$4 = A(x-2) + B(x-6)$
$4 = (A+B)x - 2A - 6B$
so that $A + B = 0$, and $-2A - 6B = 4$
Solve for $A = 1$, and $B = -1$

4. $\displaystyle\int_1^{2} \frac{4}{x^2-8x+12}\,dx = \int_1^{2} \frac{4}{(x-6)(x-2)}\,dx = \lim_{t\to 2^-}\int_1^{t} \frac{4}{(x-6)(x-2)}\,dx$

(See PFD above)

$\displaystyle = \lim_{t\to 2^-}\int_1^{t} \frac{1}{x-6} - \frac{1}{x-2}\,dx = \lim_{t\to 2^-} \ln|x-6| - \ln|x-2|\Big|_1^t$

$\displaystyle = \lim_{t\to 2^-} \ln|t-6| - \ln|t-2| - (\ln|-5| - \ln|-1|)$

$\displaystyle = \lim_{t\to 2^-} \ln 4 - \ln|t-2| - \ln 5 + 0 = \ln 4 - (-\infty) - \ln 5 = \boxed{+\infty}$ Diverges

NOTE: $\displaystyle\lim_{p\to 0^+} \ln p = -\infty$