Math 121, Section(s) 01, 02, Fall 2023

Homework #15

Due Friday, November 10th in Gradescope by 11:59 pm ET

Goal: Exploring Power Series, mainly the Interval and Radius of Convergence. Also beginning to explore the relationship between Power Series and Functions.

FIRST: Read through and understand the following Examples. Determine the Interval and Radius of Convergence. Justify.

Ex: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(5x-2)^n}{(n+5)\ 8^n}$ Use Ratio Test. $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$=\lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(5x-2)^{n+1}}{(n+6)8^{n+1}}}{\frac{\cancel{(-1)^n}(5x-2)^n}{(n+5)8^n}}\right|=\lim_{n\to\infty}\left|\frac{(5x-2)^{n+1}}{(5x-2)^n}\right|\cdot\left(\frac{n+5}{\cancel{n}+6}\right)^{1}\cdot\frac{8^n}{8^{n+1}}=\frac{|5x-2|}{8}$$

The Ratio Test gives convergence for x when $\dfrac{|5x-2|}{8}<1$ or $|5x-2|<8$.

That is $-8<5x-2<8\Longrightarrow -6<5x<10\Longrightarrow -\dfrac{6}{5}<x<2$

Manually Test Endpoints: (where $L=1$ and Ratio Test is Inconclusive)

•$x=2$ The original series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(5(2)-2)^n}{(n+5)\ 8^n}=\sum_{n=1}^{\infty}\frac{(-1)^n\cancel{8^n}}{(n+5)\ \cancel{8^n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n+5}$

which is Convergent by AST: 1. $b_n=\dfrac{1}{n+5}>0$ 2. $\lim_{n\to\infty}b_n=\lim_{n\to\infty}\dfrac{1}{n+5}=0$

3. Terms decreasing: $b_{n+1}=\dfrac{1}{n+6}<\dfrac{1}{n+5}=b_n\Rightarrow$ $x=2$ is Included in the Domain.

•$x=-\dfrac{6}{5}$ The original series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\left(5\left(-\frac{6}{5}\right)-2\right)^n}{(n+5)\ 8^n}=\sum_{n=1}^{\infty}\frac{(-1)^n(-8)^n}{(n+5)\ 8^n}$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n\cancel{8^n}}{(n+5)\cancel{8^n}}=\sum_{n=1}^{\infty}\frac{\cancel{(-1)^{2n}}^{1}}{n+5}=\sum_{n=1}^{\infty}\frac{1}{n+5}\approx\sum_{n=1}^{\infty}\frac{1}{n}$$ the Div Harmonic p-Series $p=1$.

LCT: $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{n+5}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{\cancel{n}+5}^{1}=1$ which is *Finite* and *Non-zero*. Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+5}$ is also Divergent by LCT $\Rightarrow$ $x=-\dfrac{6}{5}$ is NOT included in the Domain.

Finally, Interval of Convergence $\boxed{I=\left(-\frac{6}{5},2\right]}$ with Radius of Convergence $\boxed{R=\frac{8}{5}}$.

Ex: $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ Use Ratio Test. $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$= \lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{x^{2n+1}}\right| \frac{(2n+1)!}{(2n+3)!} = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} \to \infty\, 0 < 1$ for all x

Converges by the Ratio Test for all x and $\boxed{I = (-\infty, \infty)}$ with $\boxed{R = \infty}$.

Ex: $\displaystyle\sum_{n=0}^{\infty} n^n (x-7)^n$ Use Ratio Test. $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}(x-7)^{n+1}}{n^n(x-7)^n} \right| = \lim_{n\to\infty} \frac{(n+1)^n \to e}{n^n} (n+1)|x-7| = \infty > 1$ Diverges by the Ratio Test for all x *unless* $x - 7 = 0$ or $x = 7$. So $\boxed{I = \{7\}}$ with $\boxed{R = 0}$.

Homework: Determine the Interval and Radius of Convergence for each of the following Power Series. Use the Ratio Test and manually check convergence at the Endpoints for the Finite Intervals. Follow the examples above for statements/format for all three cases.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^4 \cdot 4^n}$
3. $\displaystyle\sum_{n=1}^{\infty} n!\, \ln n\, (x-6)^n$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, (9x-4)^n}{n^8 \cdot 5^n}$
5. $\displaystyle\sum_{n=0}^{\infty} (3n)!\, (2x-1)^n$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, (6x+1)^n}{(6n+1) \cdot 7^n}$
7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, (3x-5)^n}{(n+6)^2 \cdot 7^{n+1}}$

Find the Power Series Representation for the following functions and determine the Interval of Convergence.

9. $f(x) = \dfrac{1}{1+x}$
10. $f(x) = \dfrac{5}{1-4x}$
11. $f(x) = \dfrac{1}{3-x}$

REGULAR OFFICE HOURS

Monday: 12:00–3:00 pm

7:30–9:00 pm TA Admire, SMUDD 206

9:00–10:30 pm TA Aidee, SMUDD 206

Tuesday: 1:00–4:00 pm

6–7:30 pm TA Natalie, SMUDD 206

7:30–9:00 pm TA Gretta, SMUDD 206

9–10:30 pm TA Aidee, SMUDD 206

Wednesday: 1:00-3:00 pm

6–7:30 pm TA Admire, SMUDD 206

7:30–9:00 pm TA James, SMUDD 206

9–10:30 pm TA Natalie, SMUDD 206

Thursday: none for Professor

6:00–7:30 pm TA Gretta, SMUDD 206

7:30–9:00 pm TA James, SMUDD 206

Friday: 12:00-2:00 pm

Time for a refreshed commitment to the course for a strong finish.