

# Exam 3 Spring 2023 Answer Key

$$1 \quad \sum_{n=1}^{\infty} \frac{(-1)^n (5x+1)^n}{(5n+1) \cdot 4^n}$$

Ratio Test

Converges by Ratio Test when

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} (5x+1)^{n+1}}{(5(n+1)+1) \cdot 4^{n+1}}}{\frac{(-1)^n (5x+1)^n}{(5n+1) \cdot 4^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(5x+1)^{n+1}}{(5x+1)^n} \cdot \frac{(5n+1)}{(5n+6)} \cdot \frac{4^n}{4^{n+1}} \right| = \frac{|5x+1|}{4} < 1$$

$$\frac{|5x+1|}{4} < 1 \Rightarrow |5x+1| < 4 \Rightarrow -4 < 5x+1 < 4 \Rightarrow -5 < 5x < 3 \Rightarrow -1 < x < \frac{3}{5}$$

Manually Check Convergence at Endpoints

Take  $x = -1$ . Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n [5(-1)+1]^n}{(5n+1) \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-4)^n}{(5n+1) \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n 4^n}{(5n+1) \cdot 4^n} = \sum_{n=1}^{\infty} \frac{1}{5n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{Diverges (Harmonic) } p=1$$

$$\lim_{n \rightarrow \infty} \frac{1}{5n+1} = 0 \quad \lim_{n \rightarrow \infty} \frac{n}{5n+1} = \frac{1}{5} \quad \text{Finite, Non-Zero}$$

$\Rightarrow$  Series also Diverges by LCT

Take  $x = \frac{3}{5}$ . Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n [5(\frac{3}{5})+1]^n}{(5n+1) \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{(5n+1) \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$$

1. Isolate  $b_n = \frac{1}{5n+1} > 0$

2.  $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{5n+1} = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{5(n+1)+1} = \frac{1}{5n+6} \leq \frac{1}{5n+1} = b_n$$

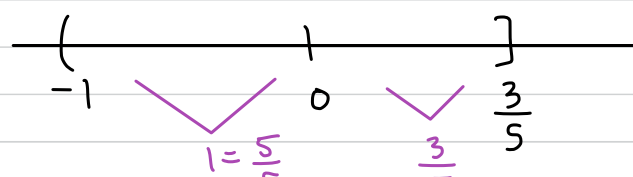
Series Converges by AST

Finally, Interval of Convergence

$$I = \left(-1, \frac{3}{5}\right]$$

Radius of Convergence

$$R = \frac{4}{5}$$



$\frac{4}{5} \rightarrow$  Half Length  $\frac{4}{5}$

$$2(a) \ln\left(1 + \frac{x^2}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^2}{4}\right)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{4^{n+1} (n+1)}$$

Need  $\left|\frac{x^2}{4}\right| < 1 \Rightarrow |x|^2 < 4 \Rightarrow |x| < 2 \Rightarrow R = 2$

$$2(b) 6x^3 \arctan(6x) = 6x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (6x)^{2n+1}}{2n+1} = 6x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n+1} x^{2n+1}}{2n+1}$$

Need  $|6x| < 1$   
 $\hookrightarrow |x| < \frac{1}{6}$   
 $-\frac{1}{6} < x < \frac{1}{6}$   
 $R = \frac{1}{6}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n+2} x^{2n+4}}{2n+1}$$

$$2(c) \frac{d}{dx} (8x^4 \sin(8x)) = \frac{d}{dx} \left( 8x^4 \sum_{n=0}^{\infty} \frac{(-1)^n (8x)^{2n+1}}{(2n+1)!} \right) = \frac{d}{dx} 8x^4 \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+2} x^{2n+5}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+2} (2n+5) x^{2n+4}}{(2n+1)!}$$

$R = \infty$  STILL After Differentiation

$$2(d) \int x^3 e^{-x^4} dx = \int x^3 \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} dx = \int x \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{n! (4n+4)} + C$$

$R = \infty$  STILL After Integration

$$\begin{aligned}
3(a) \int_0^1 x^2 \cos(x^3) dx &= \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} dx \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n)!(6n+3)} \Big|_0^1 \\
&= \frac{x^3}{1 \cdot 3} - \frac{x^9}{2! \cdot 9} + \frac{x^{15}}{4! \cdot (15)} - \dots \Big|_0^1 \\
&= \frac{1}{3} - \frac{1}{18} + \frac{1}{360} - \dots - (0 - 0 + 0 - \dots) \\
&\approx \frac{1}{3} - \frac{1}{18} = \frac{6}{18} - \frac{1}{18} = \frac{5}{18} \leftarrow \text{Estimate}
\end{aligned}$$

Using the Alternating Series Estimation Theorem (A.S.E.T)

We can estimate the full sum using only the first two terms with error at most  $\frac{1}{360} < \frac{1}{50}$  as desired

$$\begin{aligned}
3(b) \frac{1}{e} = e^{-1} &= 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \dots = \cancel{1} - \cancel{1} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \dots \\
&= \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \dots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots
\end{aligned}$$

$$\approx \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{12}{24} - \frac{4}{24} + \frac{1}{24} = \frac{9}{24} \leftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem (A.S.E.T)

We can estimate the full sum using only the first three terms with error at most  $\frac{1}{120} < \frac{1}{100}$  as desired

$$4(a) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n} (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n} \cdot \frac{\pi}{3}}{(2n+1)! \cdot \frac{\pi}{3}}$$

$$\begin{aligned}
9^n &= (3^2)^n = 3^{2n} \\
&= \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \frac{3}{\pi} \sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2\pi}
\end{aligned}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$4b. -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots = - \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right)$$

$$= - \ln(1+1) = -\ln 2$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$4c. \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \pi^{2n+1}}{4! (2n)!} = -\frac{\pi}{24} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = -\frac{\pi}{24} \cdot \cos \pi = \frac{\pi}{24}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$4d. \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} (\ln 3)^n}{5 \cdot n!} = \frac{2}{5} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (\ln 3)^n}{n!} = \frac{2}{5} \sum_{n=0}^{\infty} \frac{(-2 \ln 3)^n}{n!}$$

$$= \frac{2}{5} e^{-2 \ln 3} = \frac{2}{5} e^{\ln(3^{-2})} = \frac{2}{5} \cdot 3^{-2} = \frac{2}{5} \cdot \frac{1}{9} = \frac{2}{45}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$4e. -\frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \dots = (4 \arctan 1) - 4 = 4 \cdot \frac{\pi}{4} - 4 = \pi - 4$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

$$4 \arctan 1 = 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \dots \right) = 4 \left( -\frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \dots \right)$$

$$4(f) \pi^2 - \frac{\pi^4}{3!} + \frac{\pi^6}{5!} - \frac{\pi^8}{7!} + \dots = \pi \left( \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots \right)$$

$$= \pi \cdot \sin \pi = 0$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$5. \lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{e^{-x} - 1 + x} = \lim_{x \rightarrow 0} \frac{1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots\right)}{1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \dots - 1 + x}$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{1} - \cancel{1} + \frac{2^2 x^2}{2!} - \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} - \dots}{\cancel{1} - \cancel{x} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots - \cancel{1} + \cancel{x}}$$

Need all ...

$$= \lim_{x \rightarrow 0} \frac{\frac{2^2 x^2}{2!} - \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} - \dots}{\frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots}$$

$\frac{1}{x^2}$   
 $\frac{1}{x^2}$

$$= \lim_{x \rightarrow 0} \frac{\frac{4}{2} - \frac{2^4 x^2}{4!} + \frac{2^6 x^4}{6!} - \dots}{\frac{1}{2} - \frac{x}{3!} + \frac{x^2}{4!} - \dots} = \frac{\frac{4}{2}}{\frac{1}{2}} = 4$$



Check answer with L'Hôpital's Rule

$$\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{e^{-x} - 1 + x} \stackrel{0}{=} \lim_{x \rightarrow 0} \frac{2 \sin(2x)}{-e^{-x} + 1} \stackrel{0}{=} \lim_{x \rightarrow 0} \frac{4 \cos(2x)}{+e^{-x}} = \frac{4}{1} = 4 \text{ Match!}$$

$$6. \ln(9+x^2) = \int \frac{2x}{9+x^2} dx = \int 2x \left( \frac{1}{9+x^2} \right) dx = \int \frac{2x}{9} \left( \frac{1}{1+\frac{x^2}{9}} \right) dx = \int \frac{2x}{9} \left( \frac{1}{1-\left(-\frac{x^2}{9}\right)} \right) dx$$

$$= \int \frac{2x}{9} \sum_{n=0}^{\infty} \left( -\frac{x^2}{9} \right)^n dx = \int \frac{2x}{9} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{9^n} dx = \int 2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{9^{n+1}} dx$$

$$= 2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{9^{n+1} (2n+2)} + C = 2 \left( \frac{x^2}{9 \cdot 2} - \frac{x^4}{9^2 \cdot 4} + \frac{x^6}{9^3 \cdot 6} - \dots \right) + C$$

Test the center  $x=0$  to solve for  $+C$

$$\ln(9+0) = 2(0-0+0-\dots) + C \Rightarrow C = \ln 9$$

$$\text{Finally, } \ln(9+x^2) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{9^{n+1} (2n+2)} + \ln 9$$

### Optional Bonus #1

$$\sum_{n=0}^{\infty} \frac{n}{3^n} \text{ looks like } \sum_{n=0}^{\infty} nX^n \text{ where } X = \frac{1}{3}$$

work in reverse to find function

$$\sum_{n=0}^{\infty} nX^n = \sum_{n=0}^{\infty} n \cdot X^{n-1} \cdot X = X \sum_{n=0}^{\infty} nX^{n-1} = X \frac{d}{dX} \left( \sum_{n=0}^{\infty} X^n \right)$$

looks like derivative

$$= X \frac{d}{dX} \left( \frac{1}{1-X} \right) = X \left( \frac{1}{(1-X)^2} \right) = \frac{X}{(1-X)^2}$$

Plug in  $X = \frac{1}{3}$

$$\sum_{n=0}^{\infty} \frac{n}{3^n} = \sum_{n=0}^{\infty} n \left( \frac{1}{3} \right)^n = \frac{\frac{1}{3}}{\left(1 - \frac{1}{3}\right)^2} = \frac{\frac{1}{3}}{\frac{4}{9}} = \frac{1}{3} \cdot \frac{9}{4} = \frac{3}{4}$$

$(1-x)^{-1} = -(1-x)^{-1} \cdot (-1) = (1-x)^{-2}$

### Optional Bonus #2

$$X^3 \arctan(X^5) = X^3 \sum_{n=0}^{\infty} \frac{(-1)^n (X^5)^{2n+1}}{2n+1} = X^3 \sum_{n=0}^{\infty} \frac{(-1)^n X^{10n+5}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n X^{10n+8}}{2n+1} = \frac{X^8}{1} - \frac{X^{18}}{3} + \frac{X^{28}}{5} - \frac{X^{38}}{7} + \dots$$

$n=0$      $n=1$      $n=2$      $n=3$   
↑  $0 \cdot X^{27}$   
↑ coefficient 0

Match to General Maclaurin Series

$$f(0) + f'(0)X + \frac{f''(0)}{2!} X^2 + \dots + \frac{f^{(27)}(0)}{(27)!} X^{27} + \frac{f^{(28)}(0)}{(28)!} X^{28} + \dots$$

Equate coefficients of like Powered Terms

$$\frac{f^{(27)}(0)}{(27)!} = 0 \Rightarrow f^{(27)}(0) = 0$$

$$\frac{f^{(28)}(0)}{(28)!} = \frac{1}{5} \Rightarrow f^{(28)}(0) = \frac{(28)!}{5}$$