Exam 3 Fall 23 Answer Key

1 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (3x+5)^n}{(3n+5)^2 \cdot 4^n}$ Ratio Test

Converges by Ratio Test when

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}(3x+5)^{n+1}}{(3(n+1)+5)^2 \cdot 4^{n+1}}}{\dfrac{\cancel{(-1)^n}(3x+5)^n}{(3n+5)^2\ 4^n}} \right| = \lim_{n\to\infty} \left|\frac{(3x+5)^{n+1}}{(3x+5)^n}\right| \cdot \left(\frac{\cancel{3n+5}}{\cancel{3n+8}}\right)^2 \cdot \frac{\cancel{4^n}}{\underbrace{4^{n+1}}_{\cancel{4^n}\cdot 4}} = \frac{|3x+5|}{4} < 1$$

$|3x+5|$

$$\frac{|3x+5|}{4} < 1 \Rightarrow |3x+5| < 4 \Rightarrow -4 < 3x+5 < 4 \Rightarrow -9 < 3x < -1 \Rightarrow -3 < x < -\frac{1}{3}$$

$-5 \quad -5 \quad -5$

Manually Check Convergence at Endpoints

Take $x=-3$ Series becomes

Even

$$\sum_{n=1}^{\infty} \frac{(-1)^n [3(-3)+5]^n}{(3n+5)^2 \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-4)^n}{(3n+5)^2 \cdot 4^n} = \sum_{n=1}^{\infty} \frac{\cancel{(-1)^n(-1)^n}\ \cancel{4^n}}{(3n+5)^2 \cdot \cancel{4^n}} = \sum_{n=1}^{\infty} \frac{1}{(3n+5)^2} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\cancel{(-1)^{2n}} \to 1$

Convergent p-Series $p = 2 > 1$

Bound Terms

$$\frac{1}{(3n+5)^2} \le \frac{1}{n^2}$$

$\Rightarrow$ Series Converges by CT

OR LCT Limit

$$\lim_{n\to\infty} \frac{\dfrac{1}{(3n+5)^2}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(3n+5)^2} = \lim_{n\to\infty} \left(\frac{n}{3n+5}\right)^2 = \frac{1}{9}$$

Finite Non-Zero

$\frac{n}{3n+5} \to \frac{1}{3}$, $\left(\frac{n}{3n+5}\right)^2 \to \frac{1}{9}$

$\Rightarrow$ Series Converges by LCT

Take $x = -\frac{1}{3}$ Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\cancel{3}(-\frac{1}{\cancel{3}})+5]^n}{(3n+5)^2\ 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n\ \cancel{4^n}}{(3n+5)^2\ \cancel{4^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+5)^2} \xrightarrow{AS} \sum_{n=1}^{n} \frac{1}{(3n+5)^2}$$

Already Shown above to be Convergent using CT or LCT

ACT $\Downarrow$

Original Series Converges by ACT

OR

1. Isolate $b_n = \dfrac{1}{(3n+5)^2} > 0$

2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(3n+5)^2} = 0$ $\quad \left(\frac{1}{\infty} \to 0\right)$

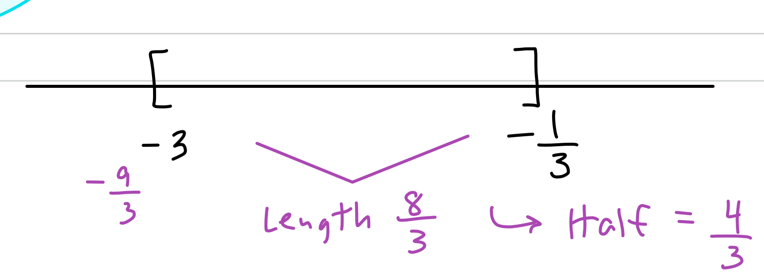
3. Terms Decreasing

$$b_{n+1} = \frac{1}{(3(n+1)+5)^2} = \frac{1}{(3n+8)^2} \le \frac{1}{(3n+5)^2} = b_n$$

Series Converges by AST

Finally, Interval of Convergence $I = \left[-3, -\frac{1}{3}\right]$

Radius of Convergence $R = \frac{4}{3}$



2(a) $\ln(1+9x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (9x^2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 9^{n+1} x^{2n+2}}{n+1}$

Need $|9x^2|<1 \Rightarrow |x|^2 < \frac{1}{9} \Rightarrow |x| < \frac{1}{3} \Rightarrow R = \frac{1}{3}$

Recall:

$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

2(b) $x^3 e^{-4x} = x^3 \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{n+3}}{n!}$

$R=\infty$

$R=\infty$

$R=\infty$

Recall:

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$R=\infty$

2(c) $\frac{d}{dx}\left(8x^4 \sin(8x)\right) = \frac{d}{dx}\left(8x^4 \sum_{n=0}^{\infty} \frac{(-1)^n (8x)^{2n+1}}{(2n+1)!}\right) = \frac{d}{dx}\, 8x^4 \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+1} x^{2n+1}}{(2n+1)!}$

$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+2} x^{2n+5}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+2} (2n+5) x^{2n+4}}{(2n+1)!}$

$R=\infty$ STILL After Differentiation

Recall:

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

2(d) $\int \frac{x^2}{8+x^3}\,dx = \int x^2\left(\frac{1}{8+x^3}\right)dx = \int \frac{x^2}{8}\left(\frac{1}{1+\frac{x^3}{8}}\right)dx = \int \frac{x^2}{8}\left(\frac{1}{1-\left(-\frac{x^3}{8}\right)}\right)dx$

$= \int \frac{x^2}{8} \sum_{n=0}^{\infty} \left(\frac{-x^3}{8}\right)^n dx = \int \frac{x^2}{8} \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{8^n}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{8^{n+1}}\,dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{8^{n+1}(3n+3)} + C$

Need $\left|\frac{-x^3}{8}\right| < 1 \Rightarrow |x|^3 < 8$

$\Rightarrow |x| < 2$

Recall: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$R=2$ STILL After Integration

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$

$$\cos\left(\frac{1}{2}\right) = 1 - \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^4}{4!} - \frac{\left(\frac{1}{2}\right)^6}{6!} + \frac{\left(\frac{1}{2}\right)^8}{8!} - \dots$$

$$= 1 - \frac{\frac{1}{4}}{\frac{2}{1}} + \frac{\frac{1}{16}}{\frac{24}{1}} - \frac{\frac{1}{64}}{\frac{720}{1}} + \dots$$

Make Sure to Connect Solution Throughout

$$= 1 - \frac{1}{8} + \frac{1}{384} - \frac{1}{46{,}080} + \dots$$

$$\approx 1 - \frac{1}{8} + \frac{1}{384} = \frac{384}{384} - \frac{48}{384} + \frac{1}{384} = \frac{337}{384}$$ ← Estimate

$$\begin{array}{r} 384 \\ -\ 48 \\ \hline 336 \\ +\ 1 \\ \hline 337 \end{array}$$

Using the Alternating Series Estimation Theorem (A.S.E.T) we can Estimate the Full Sum using only the first three terms with error at most $\frac{1}{360} < \frac{1}{50}$ as desired

4 a. $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n} \cdot \frac{\pi}{3}}{(2n+1)! \cdot \frac{\pi}{3}} = \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}$$

Flip

Recall:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \frac{3}{\pi} \underbrace{\cancel{\sin\left(\frac{\pi}{3}\right)}}_{\frac{\sqrt{3}}{2}} = \frac{3\sqrt{3}}{2\pi}$$

4b. $$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots = -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right)$$

$$= -\ln(1+1) = -\ln 2$$

Recall:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

warning: $\ln(1+(-1)) = \ln 0$ undefined won't work

4 c. $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^n (2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{2n}(2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{2}\right)^{2n}}{(2n)!} = \pi \underbrace{\cancel{\cos\left(\frac{\pi}{2}\right)}}_{0} = \pi \cdot 0 = 0$$

Recall:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

4d $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(\ln 9)^n}{2^n \cdot n!}$ Don't Drop $= - \sum_{n=0}^{\infty} \frac{(-1)^n (\ln 9)^n}{2^n n!} = - \sum_{n=0}^{\infty} \frac{\left(\frac{-\ln 9}{2}\right)^n}{n!} = -e^{\frac{-\ln 9}{2}}$

Recall:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= -e^{-\frac{1}{2}\ln 9} = -\cancel{e^{\ln}} \left(9^{-\frac{1}{2}}\right) = -\frac{1}{\sqrt{9}} = -\frac{1}{3}$$

$\frac{1}{9^{1/2}}$

Extra 1

4e. $4 + 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \ldots = 4\left(1 + \underbrace{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots}_{\text{arctan } 1}\right)$

Recall: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$

$$= 4\left[1 + \cancel{\arctan(1)}^{\frac{\pi}{4}}\right] = 4\left(1 + \frac{\pi}{4}\right) \overset{\text{OR}}{=} 4 + \pi$$

4f. $-\frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \ldots = \left(\cancel{\cos \pi}^{-1}\right) - 1 = -1 - 1 = -2$

Recall:

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \longrightarrow \cos \pi = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \ldots$

missing

5. $$\lim_{x \to 0} \frac{1 - \cos(2x)}{e^{-x} - 1 + x} = \lim_{x \to 0} \frac{1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots\right)}{1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \ldots - 1 + x}$$

$$= \lim_{x \to 0} \frac{\cancel{1} - \cancel{1} + \frac{2^2 x^2}{2!} - \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} - \ldots}{\cancel{1} - \cancel{x} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots - \cancel{1} + \cancel{x}}$$

Need all ...

$$= \lim_{x \to 0} \frac{\frac{2^2 x^2}{2!} - \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} - \ldots}{\frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

Show All Algebra Steps

$$= \lim_{x \to 0} \frac{\frac{4}{2} - \cancel{\frac{2^4 x^2}{4!} + \frac{2^6 x^4}{6!} - \ldots}^{0}}{\frac{1}{2} - \cancel{\frac{x}{3!} + \frac{x^2}{4!} - \ldots}^{0}} = \frac{\cancel{\frac{4}{2}}^{2}}{\frac{1}{2}} = 4$$

5. continued Check answer with Optional L'Hôpital's Rule

$$\lim_{x\to 0} \frac{1-\cos(2x)}{e^{-x}-1+x} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{2\sin(2x)}{-e^{-x}+1} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{4\cancel{\cos(2x)}^{1}}{+\cancel{e^{-x}}^{1}} = \frac{4}{1} = 4$$ Match!

6. $$\arctan(x^4) = \int \frac{4x^3}{1+x^8}\,dx = \int 4x^3\left(\frac{1}{1+x^8}\right)dx$$ Don't Drop dx's

$$= \int 4x^3\left(\frac{1}{1-(-x^8)}\right)dx = \int 4x^3 \sum_{n=0}^{\infty}(-x^8)^n\,dx$$

$$= \int 4x^3 \sum_{n=0}^{\infty}(-1)^n x^{8n}\,dx = \int 4\sum_{n=0}^{\infty}(-1)^n x^{8n+3}\,dx$$

$$= \cancel{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{\underbrace{8n+4}_{\cancel{4}(2n+1)}} + C = \sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{2n+1} + \cancel{C}^{0}$$

Expand in Long Form to Solve for $+C$

$$\arctan(x^4) = \frac{x^4}{1} - \frac{x^{12}}{3} + \frac{x^{20}}{5} - \ldots + C$$

Test the Center $x=0$ into both sides to solve for $+C$

$$\cancel{\arctan 0}^{0} = \cancel{0-0+0-\ldots}^{0} + C \Rightarrow C=0$$

Finally, $$\boxed{\arctan(x^4) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{2n+1}}$$ OR $$= 4\sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{8n+4}$$

Optional Check using "Substitution" into $\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$

$$\arctan(x^4) = \sum_{n=0}^{\infty}\frac{(-1)^n (x^4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{8n+4}}{2n+1}$$ Match!