Exam #2 Spring 2020

(Taken Remotely)

1a. $\int_0^5 \frac{6}{x^2-4x-5}\,dx = \int_0^5 \frac{6}{(x-5)(x+1)}\,dx = \lim_{t\to 5^-} \int_0^t \frac{6}{(x-5)(x+1)}\,dx$

Improper at $x=5$

given → PFD

$$= \lim_{t\to 5^-} \int_0^t \frac{1}{x-5} - \frac{1}{x+1}\,dx$$

$$= \lim_{t\to 5^-} \ln|x-5| - \ln|x+1| \Big|_0^t$$

$-\infty$ $0^+$ $5^-$ $|6^-|$ $-\ln 6$ $0$

$$= \lim_{t\to 5^-} \cancel{\ln|t-5|} - \cancel{\ln|t+1|} - (\ln 5 - \cancel{\ln 1})$$

Finite. Finite

$$= \boxed{-\infty} \text{ Diverges}$$

1b. $\int_6^\infty \frac{6}{x^2-4x-5}\,dx = \lim_{t\to\infty} \int_6^t \frac{6}{x^2-4x-5}\,dx = \lim_{t\to\infty} \int_6^t \frac{6}{(x-5)(x+1)}\,dx$

same PFD

$$= \lim_{t\to\infty} \int_6^t \frac{1}{x-5} - \frac{1}{x+1}\,dx$$

$$= \lim_{t\to\infty} \ln|x-5| - \ln|x+1| \Big|_6^t$$

$\infty$ $\infty$ $0$

$$= \lim_{t\to\infty} \cancel{\ln|t-5|} - \cancel{\ln|t+1|} - (\cancel{\ln 1} - \ln 7)$$

Indeterminate Difference

$$= \lim_{t\to\infty} \ln\left|\frac{t-5}{t+1}\cdot\frac{\frac{1}{t}}{\frac{1}{t}}\right| + \ln 7$$

$$= \lim_{t\to\infty} \ln\left|\frac{1-5/t}{1+1/t}\right| + \ln 7 = 0 + \ln 7 = \boxed{\ln 7} \text{ Converges}$$

1c. $\displaystyle\int_{-\infty}^{5} \frac{6}{x^2-4x+7}\,dx = \lim_{t\to-\infty} \int_t^5 \frac{6}{x^2-4x+7}\,dx$

Discriminant

$b^2-4ac = 16-4(1)(7)$

$= 16-28 = -12 < 0$

Complete Square

$\displaystyle= \lim_{t\to-\infty} \int_t^5 \frac{6}{(x-2)^2+3}\,dx$

$x^2-4x+4$

$\displaystyle= \lim_{t\to-\infty} \int_{t-2}^{3} \frac{6}{w^2+3}\,dw$

$\boxed{\begin{aligned} w &= x-2 \\ dw &= dx \end{aligned}}$

$\displaystyle= \lim_{t\to-\infty} \frac{6}{\sqrt{3}} \arctan\left(\frac{w}{\sqrt{3}}\right)\Big|_{t-2}^{3}$

$\boxed{\begin{aligned} x=t &\Rightarrow w=t-2 \\ x=5 &\Rightarrow w=5-2=3 \end{aligned}}$

$\displaystyle= \lim_{t\to-\infty} \frac{6}{\sqrt{3}} \left[\arctan\left(\frac{3}{\sqrt{3}}\right) - \arctan\left(\frac{t-2}{\sqrt{3}}\right)\right]$

$\sqrt{3}$, $\frac{\pi}{3}$, $-\infty$, $-\pi/2$

$\displaystyle= \frac{6}{\sqrt{3}}\left[\frac{\pi}{3} + \frac{\pi}{2}\right] = \frac{6}{\sqrt{3}}\left(\frac{5\pi}{6}\right) = \boxed{\frac{5\pi}{\sqrt{3}}}$ Converges

$\frac{2\pi}{6} + \frac{3\pi}{6}$

1d. $\displaystyle\int_0^e \ln x\,dx = \lim_{t\to0^+} \int_t^e \ln x\,dx = \lim_{t\to0^+} x\ln x\Big|_t^e - \int_t^e 1\,dx$

improper

$\displaystyle= \lim_{t\to0^+} x\ln x\Big|_t^e - x\Big|_t^e$

IBP

$\boxed{\begin{aligned} u &= \ln x & dv &= 1\,dx \\ du &= \tfrac{1}{x}\,dx & v &= x \end{aligned}}$

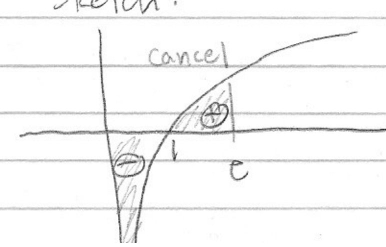
$\displaystyle= \lim_{t\to0^+} e\ln e - t\ln t - (e-t) = e - e = \boxed{0}$

$1$, $0\cdot(-\infty)$, $0$, $0$

Indeterminate

See (*)

Sketch:




(*) $\displaystyle\lim_{t\to0^+} t\cdot\ln t = \lim_{t\to0^+} \frac{\ln t}{1/t} \overset{L'H}{=} \lim_{t\to0^+} \frac{1/t}{-1/t^2} = \lim_{t\to0^+} -t = \boxed{0}$

$-\infty/\infty$

Improper Rational Function

Split-Split

$$1e.\ \int \frac{x^3+4x+1}{x^2+1}\,dx = \int x + \frac{3x+1}{x^2+1}\,dx = \frac{x^2}{2} + 3\int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

u-sub?

$$= \boxed{\frac{x^2}{2} + \frac{3}{2}\ln|x^2+1| + \arctan x + C}$$

Long Division

$$\begin{array}{r} x \\ x^2+1 \overline{)\, x^3+4x+1} \\ -(x^3+x) \\ \hline 3x+1 \end{array}$$

$1^\infty$

$$2.\ \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to\infty}\ln\left[\left(1+\frac{1}{x}\right)^x\right]}$$

Indeterminate Power

$\infty\cdot 0$ $\qquad$ $0/0$

$$= e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)} \overset{\text{Flip.}}{=} e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}}$$

$$\underset{L'H}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{1+1/x}\left(\cancel{-\frac{1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}}} = \boxed{e} \quad \text{The sequence } \boxed{\text{Converges}}$$

$$3.\ \sum_{n=1}^{\infty}\frac{(-1)^n 4^{n+1}}{3^{2n-1}} = \underset{n=1}{-\frac{4^2}{3}} + \underset{n=2}{\frac{4^3}{3^3}} - \underset{n=3}{\frac{4^4}{3^5}} + \ldots$$

$$a = -\frac{4^2}{3} = \frac{-16}{3} \qquad \text{Sum} = \frac{a}{1-r} = \frac{-16/3}{1-(-4/9)} = \boxed{\frac{-48}{13}}$$

$\frac{9}{13}$, $\frac{13}{9}$

$$r = \frac{-4}{3^2} = \frac{-4}{9}$$

4a $\sum_{n=1}^{\infty} \frac{6}{n^6} + \frac{1}{n^6+6} = \sum_{n=1}^{\infty} \frac{6}{n^6} + \sum_{n=1}^{\infty} \frac{1}{n^6+6}$

Constant Multiple of
Convergent p-Series
$p=6>1$ is Convergent

$\sum_{n=1}^{\infty} \frac{1}{n^6+6} \approx \sum_{n=1}^{\infty} \frac{1}{n^6}$ Convergent p-Series
$p=6>1$

Bound Terms
$\frac{1}{n^6+6} \leq \frac{1}{n^6} \Rightarrow$ Series Converges by CT

Sum of Two Convergent series is $\boxed{\text{Convergent}}$

4b. $\sum_{n=1}^{\infty} \frac{6}{\arctan n}$ Diverges by nTDT because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{6}{\underbrace{\arctan n}_{\to \pi/2}} = \frac{6}{\pi/2} = \frac{12}{\pi} \neq 0$$

4c. Prove $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6}$ Converges using A.C.T.

Start $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^6} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{n^6}$ Converges p-Series
$p=6>1$

$\boxed{\text{O.S. Converges by A.C.T.}}$

5a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{6n-2} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{6n-2} \sim \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges p-series (Harmonic) $p=1$

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{1}{6n-2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n\,(1/n)}{6n-2\,(1/n)} = \lim_{n\to\infty} \frac{1}{6-\cancel{2/n}_{\to 0}} = \frac{1}{6}$$ Finite Non-zero

$\Rightarrow$ A.S. also Diverges by LCT (Not A.C.)

AST on O.S.

① $b_n = \frac{1}{6n-2}$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\underbrace{6n-2}_{\to\infty}} = 0$

③ $b_{n+1} = \frac{1}{\underbrace{6(n+1)-2}_{6n+4}} \leq \frac{1}{6n-2} = b_n$

$\Rightarrow$ O.S. Converges by A.S.T.

O.S. $\boxed{C.C.}$ by definition

5b. $\sum_{n=1}^{\infty} \frac{(-1)^n n^n \cdot n!}{(2n)!}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}}(n+1)^{n+1}(n+1)!}{(2(n+1))!}}{\frac{\cancel{(-1)^n} n^n\, n!}{(2n)!}} \right| = \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{n^n}\right] \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)(2n)!}}$$

$(n+1)^n(n+1)$ $\quad (n+1)n!$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}\xrightarrow{} e \cdot \left(\frac{\cancel{n+1}}{\underbrace{2n+2}_{2(n+1)}}\right)\left(\frac{n+1\ \frac{1}{n}}{2n+1\ \frac{1}{n}}\right)$$

$$= \lim_{n\to\infty} \frac{e}{2}\left[\frac{1+\cancel{\frac{1}{n}}^{\to 0}}{2+\cancel{\frac{1}{n}}_{\to 0}}\right] = \frac{e}{4} < 1$$ O.S. $\boxed{A.C.}$ by R.T.

$\frac{1}{2}$

5c. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^3+6}{n^6+3} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{n^3+6}{n^6+3} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges p-Series $p = 3 > 1$

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{n^3+6}{n^6+3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^6+6n^3\left(\frac{1}{n^6}\right)}{n^6+3\left(\frac{1}{n^6}\right)} = \lim_{n\to\infty} \frac{1+\overbrace{\frac{6}{n^3}}^{\to 0}}{1+\underbrace{\frac{3}{n^6}}_{\to 0}} = 1$$ Finite Non-Zero

$\Rightarrow$ A.S. Converges by LCT

$\Rightarrow$ O.S. $\boxed{\text{A.C.}}$ by definition

Bonus #1

Given $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \ln 2$

$$\Rightarrow \frac{1}{2}\ln 2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \dots$$

Add Together
$$1 - \cancel{\frac{1}{2}} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cancel{\frac{1}{6}} + \frac{1}{7} - \frac{1}{8} + \dots = \ln 2$$
$$+ \quad \cancel{\frac{1}{2}} - \frac{1}{4} + \cancel{\frac{1}{6}} - \frac{1}{8} + \frac{1}{10} - \dots = \frac{1}{2}\ln 2$$

$$= 1 + \frac{1}{3} - \underbrace{\frac{2}{4}}_{-\frac{1}{2}} + \frac{1}{5} + \frac{1}{7} - \underbrace{\frac{2}{8}}_{-\frac{1}{4}} + \dots = \boxed{\frac{3}{2}\ln 2}$$