

Math 121 Exam 2 Fall 2023 Answer Key

$$1(a) \int_0^e x^2 \ln(x^2) dx = \lim_{t \rightarrow 0^+} \int_t^e x^2 \ln(x^2) dx = \lim_{t \rightarrow 0^+} \frac{x^3}{3} \cdot \ln(x^2) \Big|_t^e - \frac{2}{3} \int_t^e x^2 dx$$

$$\begin{aligned} u &= \ln(x^2) & dv &= x^2 dx \\ du &= \frac{1}{x^2} (2x) dx & v &= \frac{x^3}{3} \\ &= \frac{2}{x} dx & & \end{aligned}$$

$$\begin{aligned} &= \lim_{t \rightarrow 0^+} \frac{x^3}{3} \cdot \ln(x^2) \Big|_t^e - \frac{2x^3}{9} \Big|_t^e \\ &= \lim_{t \rightarrow 0^+} \frac{e^3}{3} \cdot \ln(e^2) - \frac{t^3}{3} \cdot \ln(t^2) - \left(\frac{2e^3}{9} - \frac{2t^3}{9} \right) \end{aligned}$$

See (*)

$$= \frac{2e^3}{3} - \frac{2e^3}{9} = \frac{6e^3}{9} - \frac{2e^3}{9} = \frac{4e^3}{9} \quad \text{Converges Match}$$

$$(*) \lim_{t \rightarrow 0^+} t^3 \cdot \ln(t^2) = \lim_{t \rightarrow 0^+} \frac{\ln(t^2)}{\frac{1}{t^3}} \stackrel{\text{L'H}}{=} \lim_{t \rightarrow 0^+} \frac{\frac{1}{2} \cdot (2t)}{\frac{-3}{t^4}} = \lim_{t \rightarrow 0^+} \frac{-2t^4}{3t} = \lim_{t \rightarrow 0^+} -\frac{2t^3}{3} = 0$$

Key Note: $\ln 0$ is undefined, so must "sneak attack" 0 using Limit 0^+

$$1(b) \int_e^{\infty} \frac{\ln x}{x^2} dx = \lim_{t \rightarrow \infty} \int_e^t (\ln x) \cdot x^{-2} dx \stackrel{\text{IBP}}{=} \lim_{t \rightarrow \infty} -\frac{\ln x}{x} \Big|_e^t + \int_e^t x^{-2} dx$$

$$\begin{aligned} u &= \ln x & dv &= x^{-2} dx \\ du &= \frac{1}{x} dx & v &= \frac{x^{-1}}{-1} = -\frac{1}{x} \end{aligned}$$

$$= \lim_{t \rightarrow \infty} -\frac{\ln x}{x} \Big|_e^t - \frac{1}{x} \Big|_e^t$$

$$= \lim_{t \rightarrow \infty} -\frac{\ln t}{t} + \frac{\ln e}{e} - \frac{1}{t} + \frac{1}{e}$$

$$= \lim_{t \rightarrow \infty} -\frac{1}{t} + \frac{1}{e} + \frac{1}{e} = \frac{2}{e} \quad \text{Converges Match!}$$

$$1(c) \int_{-\infty}^{-3} \frac{8-x}{x^2+2x+5} dx = \lim_{t \rightarrow -\infty} \int_t^{-3} \frac{8-x}{x^2+2x+5} dx \stackrel{\text{Complete Square}}{=} \lim_{t \rightarrow -\infty} \int_t^{-3} \frac{8-x}{(x+1)^2+4} dx$$

$$(x+1)^2 = x^2+2x+1$$

$$= \lim_{t \rightarrow -\infty} \int_{t+1}^{-2} \frac{8-(u-1)}{u^2+4} du = \lim_{t \rightarrow -\infty} \int_{t+1}^{-2} \frac{9-u}{u^2+4} du$$

$$\begin{cases} u = x+1 \Rightarrow x = u-1 \\ du = dx \end{cases}$$

$$\begin{cases} x = t \Rightarrow u = t+1 \\ x = -3 \Rightarrow u = -3+1 = -2 \end{cases}$$

$$= \lim_{t \rightarrow -\infty} \int_{t+1}^{-2} \frac{9}{u^2+4} - \frac{u}{u^2+4} du$$

$$= \lim_{t \rightarrow -\infty} \left[\frac{9}{2} \arctan\left(\frac{u}{2}\right) - \frac{1}{2} \ln|u^2+4| \right]_{t+1}^{-2}$$

$$= \lim_{t \rightarrow -\infty} \left[\frac{9}{2} \arctan\left(\frac{-2}{2}\right) - \frac{1}{2} \ln 8 - \left(\frac{9}{2} \arctan\left(\frac{t+1}{2}\right) - \frac{1}{2} \ln|(t+1)^2+4| \right) \right]$$

Finite Finite Finite

= ∞ Diverges
Matches! Finish all pieces Find all arctan values.

$$1(d) \int_{-4}^{-3} \frac{8-x}{x^2+2x-8} dx = \int_{-4}^{-3} \frac{8-x}{(x-2)(x+4)} dx = \lim_{t \rightarrow -4^+} \int_t^{-3} \frac{8-x}{(x-2)(x+4)} dx$$

PFD

$$\frac{8-x}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4}$$

$$= \lim_{t \rightarrow -4^+} \int_t^{-3} \left(\frac{1}{x-2} - \frac{2}{x+4} \right) dx$$

$$8-x = A(x+4) + B(x-2)$$

$$= Ax + 4A + Bx - 2B$$

$$= (A+B)x + (4A-2B)$$

Conditions

$$\bullet A+B = -1 \Rightarrow A = -1-B$$

$$\bullet 4A-2B = 8 \quad \begin{cases} 4(-1-B)-2B = 8 \\ -4-4B-2B = 8 \end{cases}$$

$$-6B = 12$$

$$B = -2$$

$$\downarrow$$

$$A = -1 - (-2) = 1$$

$$= \lim_{t \rightarrow -4^+} \left[\ln|x-2| - 2\ln|x+4| \right]_t^{-3}$$

$$= \lim_{t \rightarrow -4^+} \left[\ln|-5| - 2\ln|-1| - \left(\ln|t-2| - 2\ln|t+4| \right) \right]$$

Finite Finite blows up

= ∞ Diverges
Match! Fin Simplify $\ln 1 = 0$

$$2. \sum_{n=1}^{\infty} \frac{1}{n^2+4n+7} \rightarrow \int_1^{\infty} \frac{1}{x^2+4x+7} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2+4x+7} dx$$

Complete Square

$$\begin{aligned} u &= x+2 \\ du &= dx \end{aligned}$$

$$\begin{aligned} x=1 &\Rightarrow u=3 \\ x=t &\Rightarrow u=t+2 \end{aligned}$$

$$= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{(x+2)^2+3} dx$$

$$= \lim_{t \rightarrow \infty} \int_3^{t+2} \frac{1}{u^2+3} du$$

$$= \lim_{t \rightarrow \infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right) \Big|_3^{t+2}$$

$$= \lim_{t \rightarrow \infty} \frac{1}{\sqrt{3}} \left(\arctan\left(\frac{t+2}{\sqrt{3}}\right) - \arctan\left(\frac{3}{\sqrt{3}}\right) \right)$$

$$= \frac{1}{\sqrt{3}} \left(\frac{\pi}{2} - \frac{\pi}{3} \right) = \frac{1}{\sqrt{3}} \left(\frac{3\pi}{6} - \frac{2\pi}{6} \right) = \frac{\pi}{6\sqrt{3}} \quad \text{Integral Converges}$$

⇒ Series Converges by Integral Test

3(a) $\sum_{n=1}^{\infty} n^5+n^4+n^3+n^2+n+1$ Diverges by nTDT because

$$\lim_{n \rightarrow \infty} n^5+n^4+n^3+n^2+n+1 = \infty \neq 0$$

3(b) $\sum_{n=1}^{\infty} \frac{(n+5)^8}{\ln(n+5)}$ Diverges by nTDT because

$$\lim_{n \rightarrow \infty} \frac{(n+5)^8}{\ln(n+5)} = \lim_{x \rightarrow \infty} \frac{(x+5)^8}{\ln(x+5)} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{8(x+5)^7}{\frac{1}{x+5}} = \lim_{x \rightarrow \infty} 8(x+5)^8 = \infty \neq 0$$

3(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^8}$ A.S. $\sum_{n=1}^{\infty} \frac{1}{n^8}$ Converges p-Series $p=8 > 1$

original Series Converges by the Absolute Convergence Test (ACT)

- OR
1. Pick $b_n = \frac{1}{n^8} > 0$
 2. $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n^8} = 0$
 3. Terms Decreasing
 $b_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = b_n$

Original Series Converges by the Alternating Series Test

$$3(d) \sum_{n=1}^{\infty} \frac{\ln 5}{(n+5)^8} + \frac{(-1)^n 8}{5^{2n+1}} \stackrel{\text{split}}{=} \sum_{n=1}^{\infty} \frac{\ln 5}{(n+5)^8} + \sum_{n=1}^{\infty} \frac{(-1)^n 8}{5^{2n+1}}$$

$\sum_{n=1}^{\infty} \frac{1}{(n+5)^8} \approx \sum_{n=1}^{\infty} \frac{1}{n^8}$
 Convergent p-Series
 $p = 8 > 1$
Bound Terms
 $\frac{1}{(n+5)^8} \leq \frac{1}{n^8}$
 $\rightarrow \sum_{n=1}^{\infty} \frac{1}{(n+5)^8}$ Converges by CT
 OR LCT

$$\ln 5 \sum_{n=1}^{\infty} \frac{1}{(n+5)^8} + \left(-\frac{8}{5^3} + \frac{8}{5^5} - \frac{8}{5^7} + \dots \right)$$

Converges by CT
 Constant Multiple of Convergent Series is Convergent
 Converges by Geometric Series Test with $|r| = \left| -\frac{1}{5^2} \right| = \frac{1}{25} < 1$

Original Series Converges because the Sum of Two Convergent Series is Convergent

$$3(e) -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \dots = - \sum_{n=0}^{\infty} \frac{1}{n}$$

Note: not Alternating here

Constant Multiple of the Divergent Harmonic p-Series with $p=1$ is Divergent

$$4. \sum_{n=1}^{\infty} \frac{(-1)^n}{n^5 + n^4 + n^3 + n^2 + n + 1} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty} \frac{1}{n^5 + n^4 + n^3 + n^2 + n + 1} \approx \sum_{n=1}^{\infty} \frac{1}{n^5} \text{ Converges p-Series } p=5 > 1$$

Original Series
 Converges by the Absolute Convergence Test

Bound Terms

$$\frac{1}{n^5 + n^4 + n^3 + n^2 + n + 1} \leq \frac{1}{n^5}$$

Note: Limit Comparison also Works here

\Rightarrow Absolute Series also Converges by the Comparison Test

5(a) Diverges by nTDT using L'H Rule

Examples: $\sum_{n=2}^{\infty} \frac{n^2}{\ln n}$ $\sum_{n=2}^{\infty} \frac{e^n}{n n^n}$ $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

$\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$ $\sum_{n=1}^{\infty} \left(1 - \frac{5}{a^2}\right)^{n^2}$

Continue on to prove your case.

5(b) Converges by Comparison Test. Need "Smaller than Converge" Comparison

Examples: $\sum_{n=1}^{\infty} \frac{n^2}{n^9 + 5}$ $\sum_{n=1}^{\infty} \frac{1}{9n^2 + 5}$ $\sum_{n=1}^{\infty} \frac{3^n}{7^n + 6}$ $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^5 + 4}$ $\sum_{n=1}^{\infty} \frac{\cos^2 n}{8^n + 1}$

Sample. $\sum_{n=1}^{\infty} \frac{1}{n^6+9} \approx \sum_{n=1}^{\infty} \frac{1}{n^6}$ Converges p-Series $p=6 > 1$

Bound Terms
 $\frac{1}{n^6+9} \leq \frac{1}{n^6} \Rightarrow$ Original Series also Converges by Comparison Test

6(a) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^5+5n+8}{n^8+5} \right) \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{n^5+5n+8}{n^8+5} \approx \sum_{n=1}^{\infty} \frac{n^5}{n^8} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ Converges p-Series $p=3 > 1$

$$\lim_{n \rightarrow \infty} \frac{\frac{n^5+5n+8}{n^8+5}}{\frac{1}{n^3}} = \lim_{n \rightarrow \infty} \frac{n^8+5n^4+8n^3}{n^8+5} \cdot \frac{1}{n^8} = \lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n^4} + \frac{8}{n^5}}{1 + \frac{5}{n^8}} = 1 \quad \begin{matrix} \text{Finite} \\ \text{Non-zero} \end{matrix}$$

\Rightarrow Absolute Series also Converges by the Limit Comparison Test

\Rightarrow Original Series is Absolutely Convergent by Definition

note: no need to test original series

6(b) $\sum_{n=1}^{\infty} \frac{(-1)^n n^5 \cdot n^n \cdot n!}{(2n+1)!}$

Ratio Test

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1)^5 (n+1)^{n+1} (n+1)!}{(2(n+1)+1)!} \cdot \frac{(-1)^n n^5 n^n n!}{(2n+1)!} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^5}{n^5} \cdot \frac{(n+1)^{n+1}}{n^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n+1)!}{(2n+3)!}$$

or $\left(1 + \frac{1}{n}\right)^5$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^n}{n^n} \left(\frac{n+1}{2n+3} \right)^{\frac{1}{n}} \left(\frac{n+1}{2n+2} \right)$$

Don't Drop

$$= \lim_{n \rightarrow \infty} \frac{e}{2} \left(\frac{1 + \frac{1}{n}}{2 + \frac{3}{n}} \right)^{\frac{1}{2}} = \frac{e}{4} < 1$$

\Rightarrow the Series Converges Absolutely by the Ratio Test

$$b(c) \sum_{n=1}^{\infty} \frac{(-1)^n}{5n+8} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty} \frac{1}{5n+8} \approx \sum_{n=1}^{\infty} \frac{1}{n} \text{ Diverges } p\text{-Series } p=1$$

AST

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{5n+8}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{5n+8} \stackrel{\frac{1}{n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{5+\frac{8}{n}} = \frac{1}{5} \text{ Finite Non-Zero}$$

1. Pick $b_n = \frac{1}{5n+8} > 0$

2. $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{5n+8} = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{5(n+1)+8} \leq \frac{1}{5n+8} = b_n$$

Original Series Converges by the Alternating Series Test

⇒ Absolute Series also Diverges by the Limit Comparison Test

Original Series is

Conditionally Convergent by Definition

or show Related Function

$$f(x) = \frac{1}{5x+8} \text{ has } f'(x) = \frac{-5}{(5x+8)^2} < 0$$