Exam 1 Fall 23 Answer Key

1(a)
$$\lim_{x\to 0}\frac{\overset{0}{\ln(1+5x)}-\overset{0}{5x}}{\underset{0}{\arcsin(3x)}+\underset{1}{e^{-3x}}\underset{-1}{-1}}\ \ \frac{0}{0}\ \underset{\text{L'H}}{=}\ \lim_{x\to 0}\frac{\frac{1}{1+5x}\cdot 5-5}{\frac{1}{\sqrt{1-(3x)^2}}\cdot 3-3e^{-3x}}\ \ \frac{0}{0}$$

$$\underset{\text{prep}}{=}\ \lim_{x\to 0}\frac{5(1+5x)^{-1}-5}{3(1-9x^2)^{-1/2}-3e^{-3x}}$$

$$\underset{\text{L'H}}{=}\ \lim_{x\to 0}\frac{-5(1+5x)^{-2}\cdot 5}{\textcircled{-}\frac{3}{2}(1-9x^2)^{-3/2}(\textcircled{-}18x)\overset{--}{+}9e^{-3x}}$$

$$\underset{\text{rewrite}}{=}\ \lim_{x\to 0}\frac{\frac{-25}{\cancel{(1+5x)^2}}\xrightarrow{}-25}{\frac{27x}{\cancel{(1-9x^2)^{3/2}}}\xrightarrow{}0+\cancel{9e^{-3x}}\xrightarrow{}9\ \ 1}$$

$$=\boxed{-\frac{25}{9}}\quad \text{Match}$$

1(b)
$$\lim_{x\to 0^+}\overset{0\cdot(-\infty)}{x^3\ln x}=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^3}}\ \ \frac{-\infty}{\infty}\ \underset{\text{L'H}}{=}\ \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-3}{x^4}}\ \ \frac{-x^4}{3}\ =\lim_{x\to 0^+}\frac{-\cancel{x^3}\xrightarrow{}0}{3}=\boxed{0}\quad \text{Match}$$

$$x^{-3}\hookrightarrow -3x^{-4}$$

1(c)
$$\lim_{x\to\infty}\left(1-\arctan\left(\frac{3}{x^4}\right)\right)^{x^4}=e^{\lim_{x\to\infty}\ln\left(\left(1-\arctan\left(\frac{3}{x^4}\right)\right)^{x^4}\right)}$$

$$=e^{\lim_{x\to\infty}\overset{\infty\cdot 0}{x^4\cdot\ln\left(1-\arctan\left(\frac{3}{x^4}\right)\right)}}$$

$$=e^{\lim_{x\to\infty}\frac{\ln\left(1-\arctan\left(\frac{3}{x^4}\right)\right)}{\frac{1}{x^4}}}\ \ \frac{0}{0}$$

$$3x^{-4}\qquad \frac{1}{x^4}\to x^{-4}\longrightarrow -4x^{-5}$$

$$\underset{\text{L'H}}{=}\ e^{\lim_{x\to\infty}\frac{\cancel{\frac{1}{1-\arctan\left(\frac{3}{x^4}\right)}}\xrightarrow{}1\cdot\cancel{\frac{\textcircled{-}1}{1+\left(\frac{3}{x^4}\right)^2}}\xrightarrow{}-1\cdot\left(\frac{\cancel{-}\cancel{12}^{\,3}}{\cancel{x^5}}\right)}{\cancel{-}\frac{4}{x^5}}}\quad \cancel{-}\frac{\cancel{x^5}}{\cancel{4}}$$

$$=e^{1\cdot(-1)\cdot 3}=\boxed{e^{-3}}\quad \text{Match}$$

2. $$\int_{-3}^{3}\sqrt{9-x^2}\,dx = \int_{x=-3}^{x=3}\cancelto{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\cdot 3\cos\theta\,d\theta = 9\int_{x=-3}^{x=3}\cos^2\theta\,d\theta$$

$$\sqrt{9\underbrace{(1-\sin^2\theta)}}$$

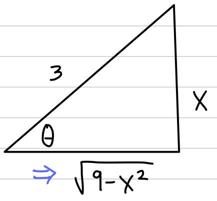
$$\sqrt{9\cos^2\theta}$$

Trig. Sub $\boxed{\begin{matrix} x = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{matrix}}$

$\sin\theta = \frac{x}{3}$

$\theta = \arcsin\left(\frac{x}{3}\right)$

$$= 9\int_{x=-3}^{x=3}\frac{1+\cos(2\theta)}{2}\,d\theta = \frac{9}{2}\int_{x=-3}^{x=3}1+\cos(2\theta)\,d\theta$$



$$= \frac{9}{2}\left(\theta + \frac{\overset{\cancel{2}\sin\theta\cos\theta}{\widehat{\sin(2\theta)}}}{\cancel{2}}\right)\Bigg|_{x=-3}^{x=3} = \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)\frac{\sqrt{9-x^2}}{3}\right)\Bigg|_{-3}^{3}$$

$$= \frac{9}{2}\left(\cancelto{\frac{\pi}{2}}{\arcsin}\left(\cancelto{1}{\frac{3}{3}}\right) + \left(\frac{3}{3}\right)\cancelto{0}{\frac{\sqrt{9-9}}{2}} - \left(\cancelto{-\frac{\pi}{2}}{\arcsin}\left(\cancelto{-1}{\frac{-3}{3}}\right) + \left(\frac{-3}{3}\right)\cancelto{0}{\frac{\sqrt{9-9}}{3}}\right)\right)$$

$$= \frac{9}{2}\left(\underbrace{\frac{\pi}{2} \overset{--}{+} \frac{\pi}{2}}_{\pi}\right) = \boxed{\frac{9\pi}{2}}$$ Match

3. $$\int_{0}^{\ln\sqrt{3}}\frac{e^{2x}}{3+e^{4x}}\,dx = \int_{0}^{\ln\sqrt{3}}\frac{e^{2x}}{3+(e^{2x})^2}\,dx = \frac{1}{2}\int_{1}^{3}\frac{1}{3+u^2}\,du = \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_{1}^{3}$$

$\boxed{\begin{matrix} u = e^{2x} \\ du = 2e^{2x}dx \\ \frac{1}{2}du = e^{2x}dx \end{matrix}}$

$\boxed{\begin{matrix} x = 0 \Rightarrow u = e^{0} = 1 \\ x = \ln\sqrt{3} \Rightarrow u = e^{2\ln\sqrt{3}} \\ = e^{\cancel{\ln}(\sqrt{3}^2)} \\ = 3 \end{matrix}}$

$$= \frac{1}{2\sqrt{3}}\left(\cancelto{\frac{\pi}{3}}{\arctan}\left(\cancelto{\sqrt{3}}{\frac{3}{\sqrt{3}}}\right) - \cancelto{\frac{\pi}{6}}{\arctan}\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{1}{2\sqrt{3}}\left(\underbrace{\frac{\pi}{3} - \frac{\pi}{6}}_{\frac{\pi}{6}}\right) = \boxed{\frac{\pi}{12\sqrt{3}}}$$ Match

4. $$\int_{1}^{e}x^3\cdot\ln x\,dx = \frac{x^4}{4}\cdot\ln x\Bigg|_{1}^{e} - \frac{1}{4}\int_{1}^{e}\cancelto{x^3}{\frac{x^4}{x}}\,dx$$

$\boxed{\begin{matrix} u = \ln x & dv = x^3dx \\ du = \frac{1}{x}dx & v = \frac{x^4}{4} \end{matrix}}$

$$= \frac{x^4}{4}\cdot\ln x\Bigg|_{1}^{e} - \frac{x^4}{16}\Bigg|_{1}^{e}$$

$$= \frac{e^4}{4}\cdot\cancelto{1}{\ln e} - \frac{1}{4}\cancelto{0}{\ln 1} - \left(\frac{e^4}{16} - \frac{1}{16}\right)$$

$$= \underset{\frac{4e^4}{16}}{\frac{e^4}{4}} - \frac{e^4}{16} \overset{--}{+} \frac{1}{16} = \frac{3e^4}{16} + \frac{1}{16} = \boxed{\frac{1+3e^4}{16}}$$ Match

5. $\int x^2 \arcsin x \, dx = \frac{x^3}{3} \arcsin x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \frac{x^3}{3}\arcsin x - \frac{1}{\ }\int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}} \cdot \cancel{\cos\theta}\,d\theta$

$\sqrt{\cos^{\cancel{2}}\theta}$

$\cancel{\cos\theta}$

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = x^2\,dx \\ du = \frac{1}{\sqrt{1-x^2}}\,dx & v = \frac{x^3}{3} \end{array}}$$

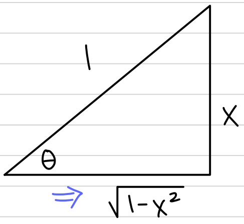
$$\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{array}}$$

$\theta = \arcsin x$

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \sin^3\theta\,d\theta$$

ODD Power

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \sin^2\theta \cdot \sin\theta\,d\theta$$



$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\int (1-\cos^2\theta) \cdot \sin\theta\,d\theta$$

⇒ ↳ $\cos\theta = \sqrt{1-x^2}$

$$= \frac{x^3}{3}\arcsin x \overset{--}{+} \frac{1}{3}\int 1 - w^2\,dw$$

$$\boxed{\begin{array}{l} w = \cos\theta \\ dw = -\sin\theta\,d\theta \\ -dw = \sin\theta\,d\theta \end{array}}$$

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\left(w - \frac{w^3}{3}\right) + C$$

$$= \frac{x^3}{3}\arcsin x + \frac{1}{3}\left(\cos\theta - \frac{1}{3}\cos^3\theta\right) + C$$

$$= \boxed{\frac{x^3}{3}\arcsin x + \frac{1}{3}\left(\sqrt{1-x^2} - \frac{1}{3}\left(\sqrt{1-x^2}\right)^3\right) + C}$$

OR $(1-x^2)^{3/2}$

6. $\int \frac{x}{(x^2+4)^{7/2}}\,dx = \int \frac{x}{\left(\sqrt{x^2+4}\right)^7}\,dx = \int \frac{2\tan\theta}{\left(\sqrt{4\tan^2\theta+4}\right)^{⑦}} \cdot 2\sec^2\theta\,d\theta$ Don't Drop

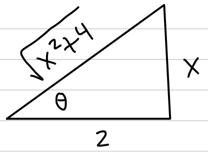
$\sqrt{4(\tan^2\theta+1)}$

$\sqrt{4\sec^2\theta}$

Must Use Trig. Sub here

$$\boxed{\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\,d\theta \end{aligned}}$$

$\tan\theta = \frac{x}{2}$



$\hookrightarrow \cos\theta = \frac{2}{\sqrt{x^2+4}}$

$$= \int \frac{2\tan\theta}{\left(\sqrt{4\sec^2\theta}\right)^7} \cdot 2\sec^2\theta\,d\theta$$

$(2\sec\theta)^7$

$$= \frac{2^2}{2^7}\int \frac{\tan\theta}{\cancel{\sec^7}\theta}\ \cancel{\sec^2}\theta\,d\theta$$

$\sec^5\theta$

$$= \frac{1}{2^5}\int \tan\theta\cdot\cos^5\theta\,d\theta$$

$$= \frac{1}{32}\int \frac{\sin\theta}{\cancel{\cos\theta}}\cdot \overset{\cos^4\theta}{\cancel{\cos^5}}\theta\,d\theta$$

$$= \frac{1}{32}\int \sin\theta\cdot\cos^4\theta\,d\theta$$ u-sub

$$\boxed{\begin{aligned} u &= \cos\theta \\ du &= -\sin\theta\,d\theta \\ -du &= \sin\theta\,d\theta \end{aligned}}$$

don't drop

$$= -\frac{1}{32}\int u^4\,du$$

$$= -\frac{1}{32}\cdot\left(\frac{u^5}{5}\right) + C$$

$$= \boxed{-\frac{1}{32}\cdot\frac{1}{5}\cdot\left(\frac{2}{\sqrt{x^2+4}}\right)^5 + C}$$

OR

$$= -\frac{1}{\cancel{32}}\cdot\frac{1}{5}\cdot\frac{\cancel{32}}{\left(\sqrt{x^2+4}\right)^5} + C$$

$$= \boxed{-\frac{1}{5(x^2+4)^{\frac{5}{2}}} + C}$$ Match

note: could run u-sub to confirm this answer, not required