Approximations using only a few terms of the MacLaurin Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \frac{1}{(10)!} + \frac{1}{(11)!} + \frac{1}{(12)!} + \dots$$

$$e = 2.7182818284590452353602874713 5\dots$$

$$e \approx 1 + 1 = 2$$

$$e \approx 1 + 1 + \frac{1}{2!} = 2.5$$

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708333\dots$$

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \frac{1}{(10)!} = 2.7182818011\dots$$

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \frac{1}{(10)!} + \dots + \frac{1}{(19)!} + \frac{1}{(20)!} = 2.71828182845904523533978449\dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

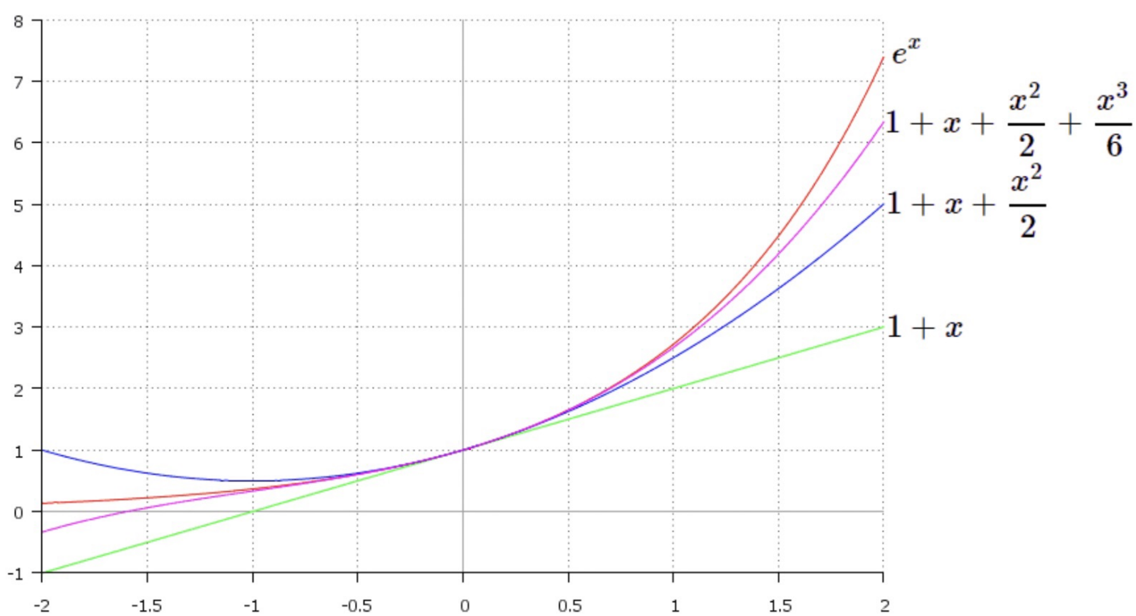
$$e^{\frac{1}{2}} = 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{\left(\frac{1}{2}\right)^4}{4!} + \dots$$

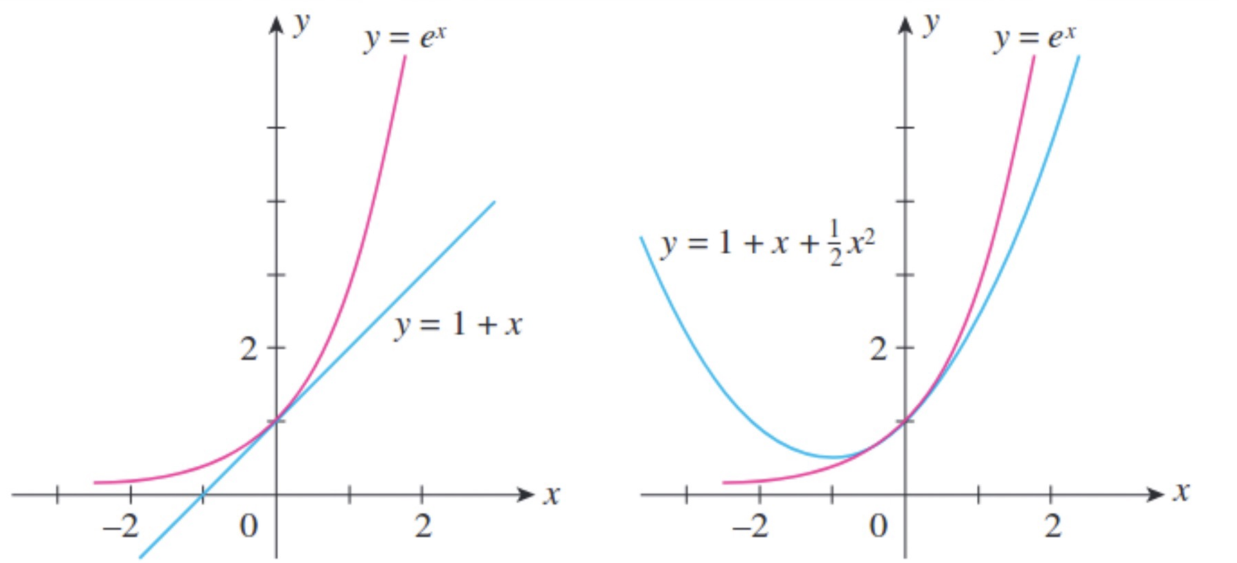
$$\sqrt{e} = 1.648721270700128146\dots$$

$$e^{\frac{1}{2}} \approx 1 + \frac{1}{2} = 1.5$$

$$e^{\frac{1}{2}} \approx 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} = 1.625$$

$$e^{\frac{1}{2}} \approx 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} = 1.64583\overline{3}$$





y
y = e^x
y = 1 + x
2
x
−2
0
2
y
y = e^x
y = 1 + x + ½x²
2
x
−2
0
2

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots$$

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} + \ldots$$

$$\boxed{\sin 1 = 0.841570984807\ldots}$$

$\sin 1 \approx 1$

$\sin 1 \approx 1 - \frac{1}{3!} = \frac{5}{6} = 0.8\overline{333}$ with error at most $\frac{1}{5!} = \frac{1}{120} = 0.008\overline{33}$ by ASET

$\sin 1 \approx 1 - \frac{1}{3!} + \frac{1}{5!} = \frac{101}{120} = 0.841\overline{666}$ with error at most $\frac{1}{7!} = \frac{1}{5040} = 0.00019841269\ldots$ by ASET

$\sin 1 \approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} = \frac{4241}{5040} = 0.84146825\ldots$ with error at most $\frac{1}{9!} = \frac{1}{362880} = 0.0000027557\ldots$ by ASET

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots$$

$$\sin 0.16 = 0.16 - \frac{(0.16)^3}{3!} + \frac{(0.16)^5}{5!} - \frac{(0.16)^7}{7!} + \frac{(0.16)^9}{9!} + \ldots$$

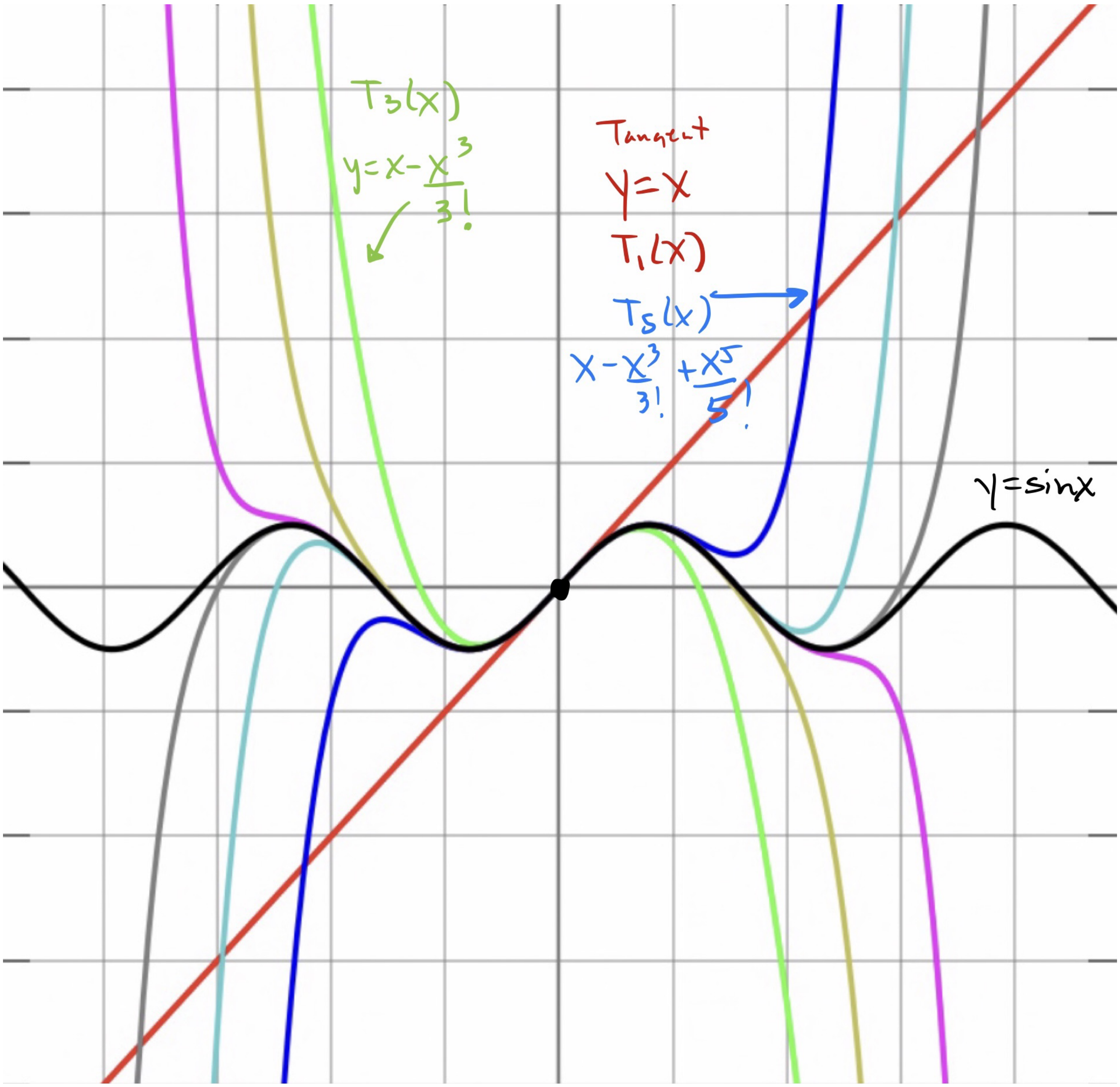
$$\boxed{\sin 0.16 = 0.15931820661424596331\,1463\ldots}$$

$\sin 0.16 \approx 0.16$

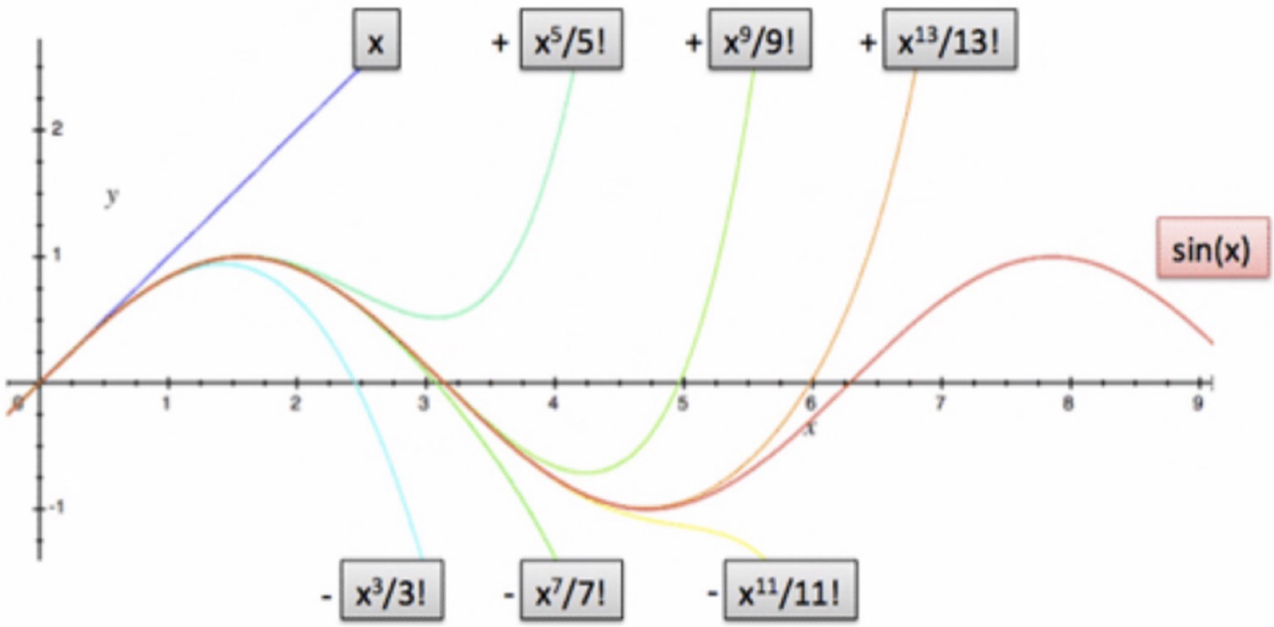
$\sin 0.16 \approx 0.16 - \frac{(0.16)^3}{3!} = 0.15931733\ldots$ with error at most $\frac{(0.16)^5}{5!} = 0.000000873\ldots$ by ASET

$\sin 0.16 \approx 0.16 - \frac{(0.16)^3}{3!} + \frac{(0.16)^5}{5!} = 0.1593182071466\ldots$

with error at most $\frac{(0.16)^7}{7!} \approx 5.326 * 10^{-10}$ by ASET y=sin x



Better Models of Sine



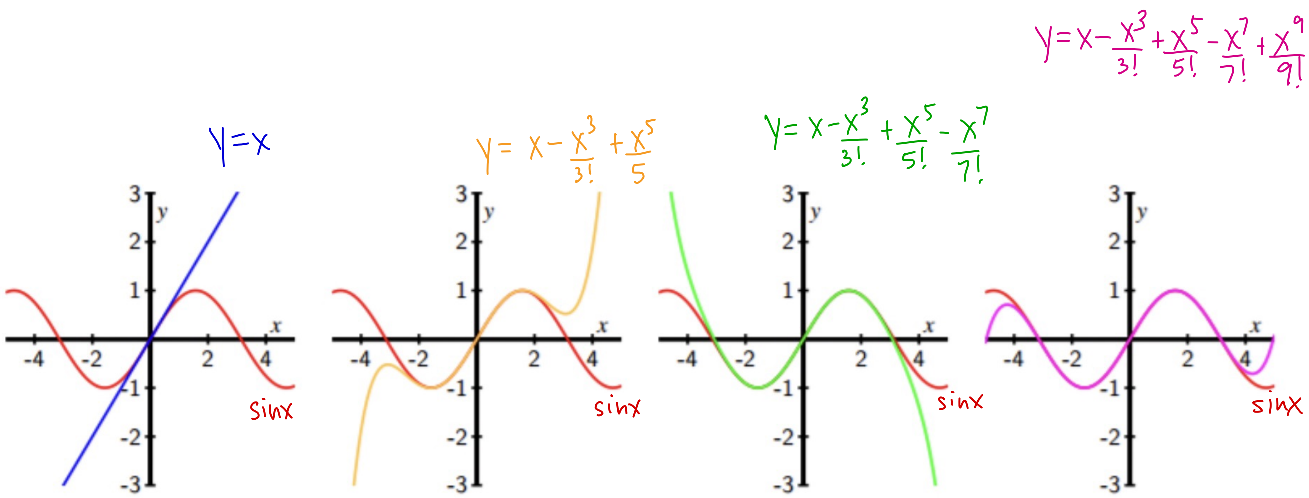
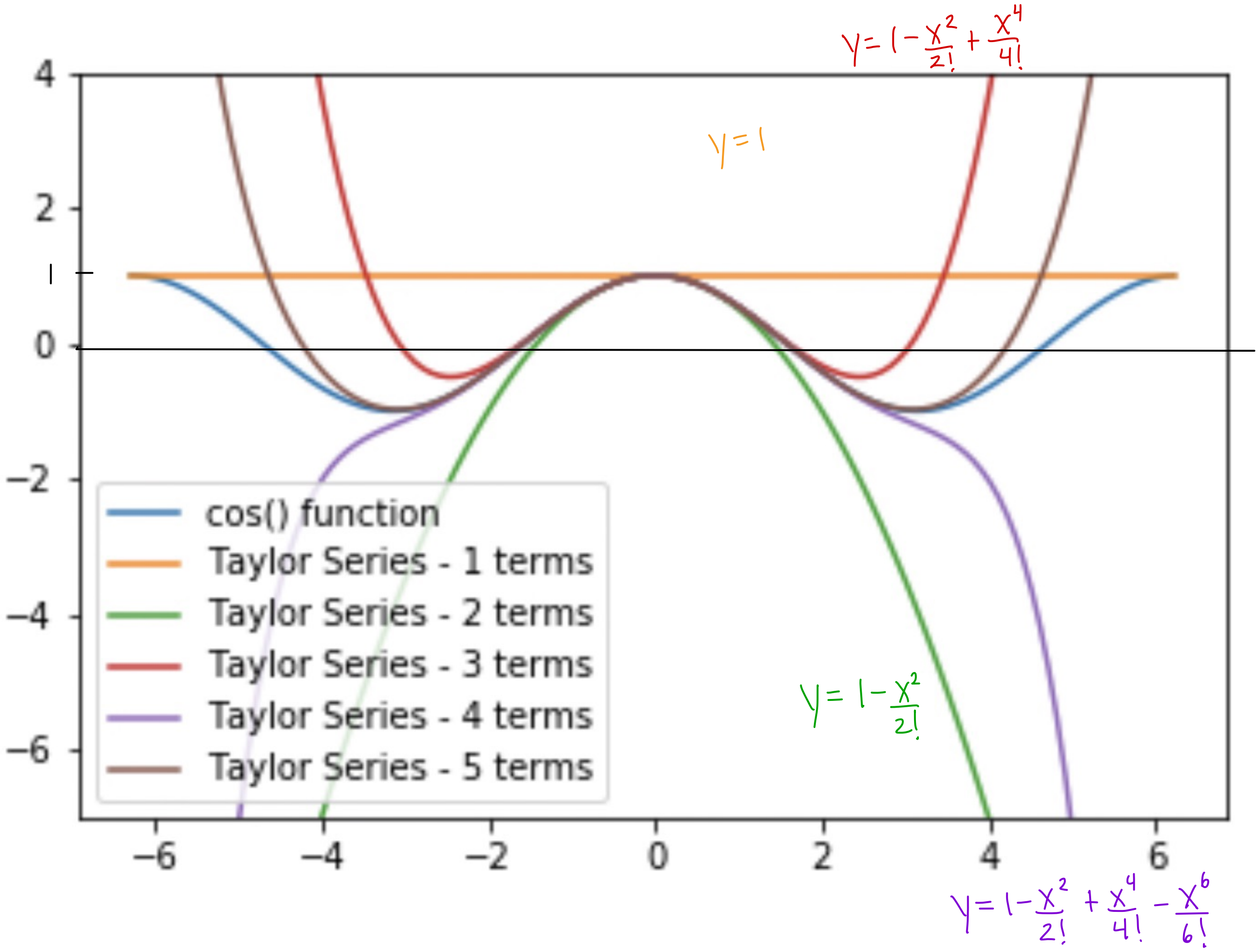
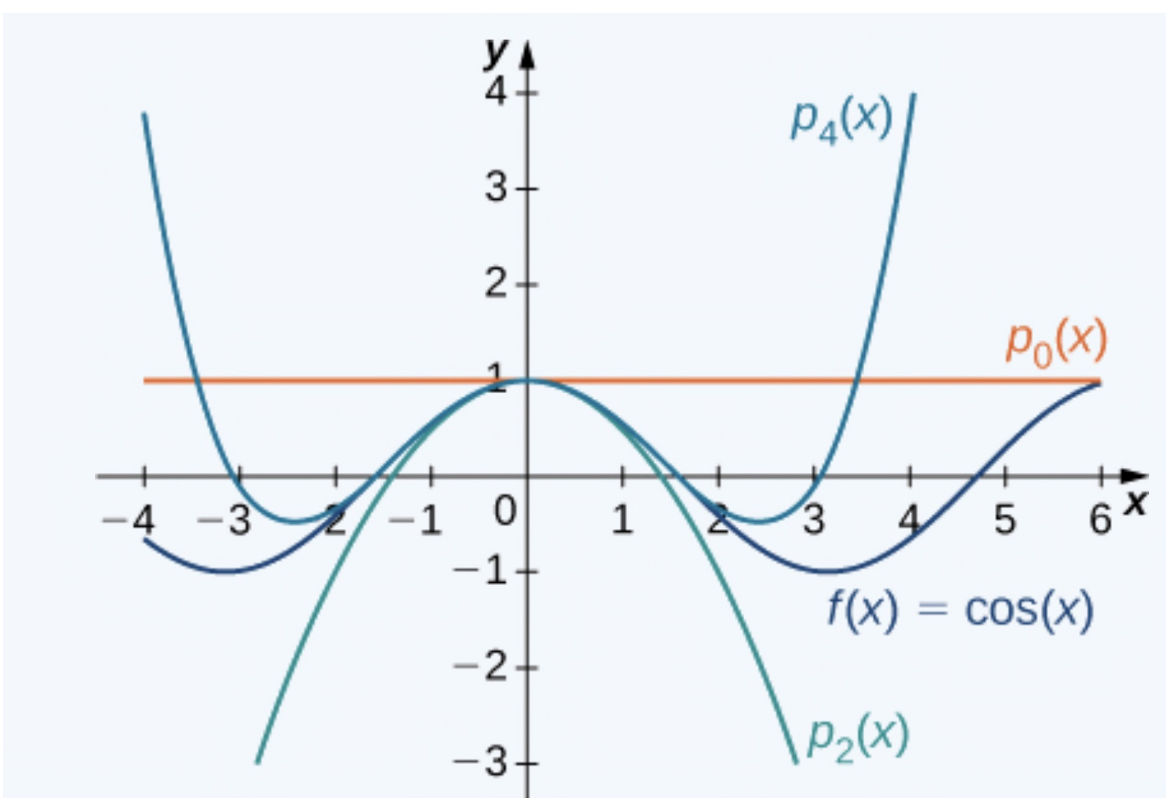


Figure* 8.5.1*:* *The order 1, 5, 7, and 9 Taylor polynomials centered at $x = 0$ for $f(x) = \sin(x)$.



cos() function
Taylor Series - 1 terms
Taylor Series - 2 terms
Taylor Series - 3 terms
Taylor Series - 4 terms
Taylor Series - 5 terms
$y=1$
$y=1-\frac{x^2}{2!}$
$y=1-\frac{x^2}{2!}+\frac{x^4}{4!}$
$y=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$



$p_4(x)$
$p_0(x)$
$f(x) = \cos(x)$
$p_2(x)$

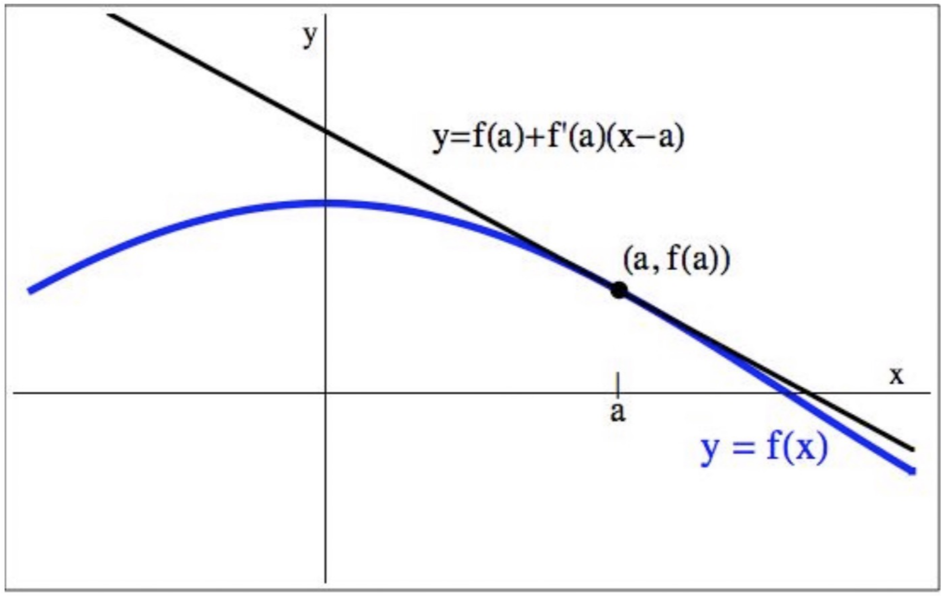


FIGURE 1. The line $y = f(a) + f'(a)(x-a)$ tangent to the graph of y = f(x) at the point (a, f(a)).

Example 1.2

Find the Taylor polynomial of degree three for f(x) = sin x, centered at x = $5\pi/6$.

Solution:

$$
\begin{aligned}
f(x) &= \sin x, & f(5\pi/6) &= 1/2, \\
f'(x) &= \cos x, & f'(5\pi/6) &= -3/2, \\
f''(x) &= -\sin x, & f''(5\pi/6) &= -1/2, \\
f(x) &= -\cos x, & f'''(5\pi/6) &= \sqrt{3}/2,
\end{aligned}
$$

The Taylor polynomial of degree three (the cubic that best fits y = sin x near x = $5\pi/6$) is

$$
\begin{aligned}
T_3(x) &= f(5\pi/6) + f'(5\pi/6)(x - 5\pi/6) + \frac{f''(5\pi/6)}{2}\left(x - 5\pi/6\right)^2 + \frac{f'''(5\pi/6)}{6}\left(x - 5\pi/6\right)^3 = \\
&= \frac{1}{2} - \frac{\sqrt{3}}{2}(x - 5\pi/6) - \frac{1}{4}\left(x - 5\pi/6\right)^2 + \frac{\sqrt{3}}{12}\left(x - 5\pi/6\right)^3.
\end{aligned}
$$

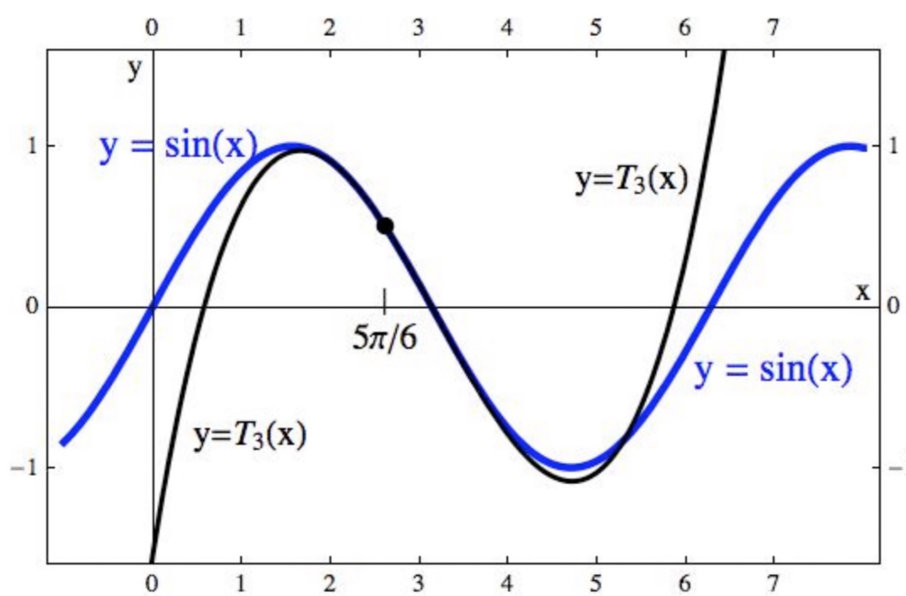


FIGURE 3. The Taylor polynomial $T_3(x)$ for f(x) = sin x, centered at x = $5\pi/6$.