Review Handout for Derivatives and u-Substitution-Math 121

Compute the following derivatives. Do not worry about simplifications here.

1. $y = \sin^3(x^3) \qquad y' = 3\sin^2(x^3)\cdot\cos(x^3)\cdot 3x^2 = 9x^2\sin^2(x^3)\cos(x^3)$

2. $y = \cos^2(3x) \qquad y' = 2\cos(3x)\cdot(-\sin(3x))\cdot 3 = -6\cos(3x)\sin(3x)$

3. $f(t) = t^2\sin^5(2t)$
   $f'(t) = t^2[5\sin^4(2t)\cdot\cos(2t)\cdot 2] + \sin^5(2t)\cdot 2t = 10t^2\sin^4(2t)\cos(2t) + 2t\sin^5(2t)$

4. $H(x) = \left(1 - \dfrac{2}{x^2}\right)^5 \qquad H'(x) = 5\Big(1 - \dfrac{2}{x^2}\Big)^4\cdot(4x^{-3}) = \dfrac{20}{x^3}\Big(1 - \dfrac{2}{x^2}\Big)^4$

5. $f(x) = \sqrt[3]{x^3+8} \qquad f'(x) = \dfrac{1}{3}(x^3+8)^{-2/3}\cdot 3x^2 = \dfrac{x^2}{(x^3+8)^{2/3}}$

6. $g(t) = \dfrac{t^3 + \tan\left(\dfrac{1}{t}\right)}{1+t^2}$

$$g'(t) = \frac{(1+t^2)\Big[3t^2 + \sec^2\left(\frac{1}{t}\right)\cdot\left(-\frac{1}{t^2}\right)\Big] - \Big(t^3 + \tan\left(\frac{1}{t}\right)\Big)\cdot(2t)}{(1+t^2)^2}$$

$$\ldots = \frac{t^4 + 3t^2 - (1+t^{-2})\sec^2\Big(\frac{1}{t}\Big) - 2t\tan\Big(\frac{1}{t}\Big)}{(1+t^2)^2}$$

7. $p(x) = \dfrac{1}{(-2x+3)^5} \qquad p'(x) = -5(-2x+3)^{-6}\cdot(-2) = \dfrac{10}{(-2x+3)^{-6}}$

8. $r(x) = \dfrac{(2x+1)^3}{(3x+1)^4}$

$$r'(x) = \frac{(3x+1)^4\cdot 3(2x+1)^2\cdot(2) - (2x+1)^3\cdot 4(3x+1)^3\cdot(3)}{(3x+1)^8}$$

$$= \frac{6(2x+1)^2(3x+1)^3[(3x+1) - 2(2x+1)]}{(3x+1)^8} = \frac{6(2x+1)^2[3x+1-4x-2]}{(3x+1)^5}$$

$$= \frac{6(2x+1)^2(-x-1)}{(3x+1)^5} = \frac{-6(x+1)(2x+1)^2}{(3x+1)^5}$$

9. $S(x) = \left(\dfrac{1+2x}{1+3x}\right)^4$

$$S'(x) = 4\Big(\frac{1+2x}{1+3x}\Big)^3\cdot\Big[\frac{(1+3x)\cdot 2 - (1+2x)\cdot 3}{(1+3x)^2}\Big]$$

$$= \frac{4(1+2x)^3\cdot[2+6x-3-6x]}{(1+3x)^5} = \frac{-4(1+2x)^3}{(1+3x)^5}$$

10. $g(x) = \cos(3x)\sin(4x)$
    $g'(x) = \cos(3x)\cos(4x)\cdot 4 + \sin(4x)\big(-\sin(3x)\big)\cdot(3) = 4\cos(3x)\cos(4x) - 3\sin(3x)\sin(4x)$

11. $g(x) = \dfrac{\cos(3x)}{\sin(4x)}$

$$g'(x) = \frac{\sin(4x)\big(-\sin(3x)\cdot 3\big) - \cos(3x)\cos(4x)\cdot 4}{\sin^2(4x)} = \frac{-3\sin(3x)\sin(4x) - 4\cos(3x)\cos(4x)}{\sin^2(4x)}$$

12. $y = \big((x^2+3x)^4 + x\big)^{-\frac{5}{7}}$

$$y' = -\frac{5}{7}\big[(x^2+3x)^4 + x\big]^{-12/7}[4(x^2+3x)^3(2x+3)+1]$$

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Compute the following integrals.

1. $$\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \tan(3x)\,dx = \int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \frac{\sin(3x)}{\cos(3x)}\,dx = -\frac{1}{3}\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{1}{u}\,du = -\frac{1}{3}\ln|u|\Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} = -\frac{1}{3}\left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= -\frac{1}{3}\left(\ln\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right) = -\frac{1}{3}\left(\ln\left(\frac{1}{\sqrt{3}}\right)\right) = -\frac{1}{3}\left(\ln 1 - \ln\sqrt{3}\right) = -\frac{1}{3}\left(0 - \ln\sqrt{3}\right) = \boxed{\frac{\ln\sqrt{3}}{3}}$$

$\overset{\text{or}}{=} \boxed{\dfrac{\ln 3}{6}}$ Here $\boxed{\begin{aligned} u &= \cos(3x) \\ du &= -\sin(3x)(3)\ dx \\ -\tfrac{1}{3}du &= \sin(3x)\ dx \end{aligned}}$ and $\boxed{\begin{aligned} x = \tfrac{\pi}{18} &\implies u = \cos\left(\tfrac{3\pi}{18}\right) = \cos\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2} \\ x = \tfrac{\pi}{9} &\implies u = \cos\left(\tfrac{3\pi}{9}\right) = \cos\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2} \end{aligned}}$

2. $$\int_e^{e^4} \frac{3}{x\sqrt{\ln x}}\,dx = 3\int_1^4 \frac{1}{\sqrt{u}}\,du = 3\int_1^4 u^{-\frac{1}{2}}\,du = 6\sqrt{u}\Big|_1^4 = 6\left(\sqrt{4} - \sqrt{1}\right) = 6(2-1) = \boxed{6}$$

Here $\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\ dx \end{aligned}}$ and $\boxed{\begin{aligned} x = e &\implies u = \ln e = 1 \\ x = e^4 &\implies u = \ln e^4 = 4 \end{aligned}}$

3. $$\int e^{x^2+\ln x+1}\,dx = \int e^{x^2}e^{\ln x}e\,dx = e\int xe^{x^2}\,dx = \frac{e}{2}\int e^u\,du = \frac{e}{2}e^u + C = \boxed{\left(\frac{e}{2}\right)e^{x^2} + C}$$

or $= \boxed{\dfrac{e^{x^2+1}}{2} + C}$ Here $\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\ dx \\ \tfrac{1}{2}\ du &= x\ dx \end{aligned}}$

4. $$\int_0^{\ln 3} \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\int_2^{10} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_2^{10} = \frac{1}{2}\left(\ln|u|\Big|_2^{10}\right) = \frac{1}{2}(\ln(10) - \ln 2)$$

$$= \frac{1}{2}\left(\ln\left(\frac{10}{2}\right)\right) = \boxed{\frac{\ln 5}{2}}$$

Here $\boxed{\begin{aligned} u &= 1+e^{2x} \\ du &= 2e^{2x}dx \\ \tfrac{1}{2}du &= e^{2x}dx \end{aligned}}$ and $\boxed{\begin{aligned} x = 0 &\implies u = 1+e^0 = 1+1 = 2 \\ x = \ln 3 &\implies u = 1+e^{2\ln 3} = 1+e^{\ln(3^2)} = 1+e^{\ln(9)} = 1+9 = 10 \end{aligned}}$

5. $$\int_0^{\frac{\pi}{6}} \frac{\cos x}{(1+6\sin x)^2}\ dx = \frac{1}{6}\int_{u=1}^{u=4} u^{-2}\ du = -\frac{1}{6}u^{-1}\Big|_1^4 = -\frac{1}{24}+\frac{1}{6} = \frac{3}{24} = \boxed{\frac{1}{8}}$$

Here $\boxed{\begin{aligned} u &= 1+6\sin x \\ du &= 6\cos x dx \\ \frac{1}{6}du &= \cos x dx \end{aligned}}$ and $\boxed{\begin{aligned} x=0 &\implies u=1 \\ x=\tfrac{\pi}{6} &\implies u = 1+6\sin(\tfrac{\pi}{6}) = 1+6(\tfrac{1}{2}) = 4 \end{aligned}}$

6. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin(2x)\ dx = \frac{1}{2}\int_{u=\frac{\pi}{2}}^{u=\frac{2\pi}{3}} \sin u\ du = -\frac{1}{2}\cos u\Big|_{\frac{\pi}{2}}^{\frac{2\pi}{3}} = -\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{2} = -\frac{1}{2}\left(-\frac{1}{2}\right) = \boxed{\frac{1}{4}}$$

Here $\boxed{\begin{aligned} u &= 2x \\ du &= 2dx \\ \frac{1}{2}du &= dx \end{aligned}}$ and $\boxed{\begin{aligned} x=\tfrac{\pi}{4} &\implies u=\tfrac{\pi}{2} \\ x=\tfrac{\pi}{3} &\implies u=\tfrac{2\pi}{3} \end{aligned}}$

7. $$\int_0^{\pi} \sin^2\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right)\ dx = 6\int_{u=0}^{u=\frac{1}{2}} u^2\ du = 2u^3\Big|_0^{\frac{1}{2}} = \boxed{\frac{1}{4}}$$

Here $\boxed{\begin{aligned} u &= \sin\left(\tfrac{x}{6}\right) \\ du &= \tfrac{1}{6}\cos\left(\tfrac{x}{6}\right) dx \\ 6du &= \cos\left(\tfrac{x}{6}\right) dx \end{aligned}}$ and $\boxed{\begin{aligned} x=0 &\implies u=0 \\ x=\pi &\implies u=\tfrac{1}{2} \end{aligned}}$

8. $$\int_0^{\frac{\pi}{8}} \tan^3(2x)\sec^2(2x)\ dx = \frac{1}{2}\int_{u=0}^{u=1} u^3\ du = \frac{1}{2}\frac{u^4}{4}\Big|_0^1 = \frac{1}{8} - 0 = \boxed{\frac{1}{8}}$$

Here $\boxed{\begin{aligned} u &= \tan(2x) \\ du &= 2\sec^2(2x)dx \\ \tfrac{1}{2}du &= \sec^2(2x)dx \end{aligned}}$ and $\boxed{\begin{aligned} x=0 &\implies u=0 \\ x=\tfrac{\pi}{8} &\implies u=\tan\tfrac{\pi}{4}=1 \end{aligned}}$

9. $$\int \sqrt{x}\cos(x\sqrt{x})\ dx = \frac{2}{3}\int \cos u\ du = \frac{2}{3}\sin u + C = \boxed{\frac{2}{3}\sin(x^{\frac{3}{2}}) + C}$$

Here $\boxed{\begin{aligned} u &= x^{\frac{3}{2}} \\ du &= \tfrac{3}{2}x^{\frac{1}{2}}dx \\ \tfrac{2}{3}du &= \sqrt{x}dx \end{aligned}}$

10. $$\int \frac{1}{t^2}\sin\left(\frac{1}{t}\right)\ dt = -\int \sin u\ du = \cos u + C = \boxed{\cos\tfrac{1}{t} + C}$$

Here $\boxed{\begin{aligned} u &= \tfrac{1}{t} \\ du &= -\tfrac{1}{t^2}dt \\ -du &= \tfrac{1}{t^2}dt \end{aligned}}$

11. $\int x(x+1)^{14}\ dx = \int (u-1)u^{14}\ du = \int u^{15} - u^{14}\ du = \frac{1}{16}u^{16} - \frac{1}{15}u^{15} + C$

$= \boxed{\frac{(x+1)^{16}}{16} - \frac{(x+1)^{15}}{15} + C}$ Here $\boxed{\begin{aligned} u &= x+1 \Longrightarrow x = u-1 \\ du &= dx \end{aligned}}$

Sometimes called an *inverted* substitution.

12. $\int 7\cos(5x) - 5\sin(7x)\ dx = \frac{7}{5}\int \cos u\ du - \frac{5}{7}\int \sin w\ dw = \frac{7}{5}\sin u + \frac{5}{7}\cos w + C =$
$\boxed{\frac{7}{5}\sin(5x) + \frac{5}{7}\cos(7x) + C}$

Here $\boxed{\begin{aligned} u &= 5x \\ du &= 5dx \\ \tfrac{1}{5}du &= dx \end{aligned}}$ and $\boxed{\begin{aligned} w &= 7x \\ dw &= 7dx \\ \tfrac{1}{7}dw &= dx \end{aligned}}$

13. $\int x\sqrt{2-3x^2}\ dx = -\frac{1}{6}\int \sqrt{w}\ dw = -\frac{1}{6}\left(\frac{2}{3}\right) w^{\frac{3}{2}} + C = -\frac{1}{9}w^{\frac{3}{2}} + C = \boxed{-\frac{1}{9}(2-3x^2)^{\frac{3}{2}} + C}$

Here $\boxed{\begin{aligned} w &= 2-3x^2 \\ dw &= -6xdx \\ -\tfrac{1}{6}dw &= xdx \end{aligned}}$

14. $\int \frac{e^{-\frac{1}{x}}}{7x^2}\ dx = \frac{1}{7}\int e^u\ du = \frac{1}{7}e^u + C = \boxed{\frac{1}{7}e^{-\frac{1}{x}} + C}$

Here $\boxed{\begin{aligned} u &= -\tfrac{1}{x} \\ du &= \tfrac{1}{x^2}dx \end{aligned}}$

15. $\int \frac{e^x}{(e^x-1)^2}\ dx = \int \frac{1}{u^2}\ du = -u^{-1} + C = \boxed{-\frac{1}{e^x-1} + C}$

Here $\boxed{\begin{aligned} u &= e^x - 1 \\ du &= e^x dx \end{aligned}}$

16. $\int \frac{3e^{7x}}{\sqrt{1-e^{7x}}}\ dx = -\frac{3}{7}\int \frac{1}{\sqrt{w}}\ dw = -\frac{3}{7}(2)w^{\frac{1}{2}} + C = -\frac{6}{7}\sqrt{w} + C = \boxed{-\frac{6}{7}\sqrt{1-e^{7x}} + C}$

Here $\boxed{\begin{aligned} w &= 1-e^{7x} \\ dw &= -7e^{7x}dx \\ -\tfrac{1}{7}dw &= e^{7x}dx \end{aligned}}$

Caution: For u-substitution definite integrals, I will be changing my limits of integration (like your book does). If you choose not to change your limits of integration, then you **must** mark your original limits carefully. Please do not mix and match the two variables. See me with questions...