Math 121 Self-Assessment Quiz #4

Answer Key

- Please see the course webpage for the answer key.

1. Compute $\displaystyle\int \frac{x^4-2x^3+4x^2-17x+10}{x^3-3x^2+2x-6}\,dx = \int \frac{x^4-2x^3+4x^2-17x+10}{(x-3)(x^2+2)}\,dx$

$$= \int x+1+\frac{5x^2-13x+16}{(x-3)(x^2+2)}\,dx \overset{\text{PFD}}{=} \int x+1+\frac{2}{x-3}+\frac{3x-4}{x^2+2}\,dx$$

$$\int x+1+\frac{2}{x-3}+\frac{3x}{x^2+2}-\frac{4}{x^2+2}\,dx$$

$$= \boxed{\frac{x^2}{2}+x+2\ln|x-3|+\frac{3}{2}\ln|x^2+2|-\frac{4}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+C}$$

Long division yields:

$$\begin{array}{r} x+1 \\ x^3-3x^2+2x-6\overline{)x^4-2x^3+4x^2-17x+10} \\ \underline{-(x^4-3x^3+2x^2-6x)} \\ x^3+2x^2-11x+10 \\ \underline{-(x^3-3x^2+2x-6)} \\ 5x^2-13x+16 \end{array}$$

Partial Fractions Decomposition:

$$\frac{5x^2-13x+16}{(x-3)(x^2+2)} = \frac{A}{x-3}+\frac{Bx+C}{x^2+2}$$

Clearing the denominator yields:

$5x^2-13x+6 = A(x^2+2)+(Bx+C)(x-3)$
$5x^2-13x+6 = (A+B)x^2+(C-3B)x+2A-3C$
so that $A+B=5,\ C-3B=-13$ and $2A-3C=16$
Solve for $A=2,\ B=3$ and $C=-4$

2. Compute $\displaystyle\int \frac{x+7}{x^2+6x+14}\,dx = \int \frac{x+7}{(x+3)^2+5}\,dx$ complete the square

$$= \int \frac{w+4}{w^2+5}\,dw = \int \frac{w}{w^2+5}+\frac{4}{w^2+5}\,dw = \frac{1}{2}\ln|w^2+5|+\frac{4}{\sqrt{5}}\arctan\left(\frac{w}{\sqrt{5}}\right)+C$$

$$= \boxed{\frac{1}{2}\ln|(x+3)^2+5|+\frac{4}{\sqrt{5}}\arctan\left(\frac{x+3}{\sqrt{5}}\right)+C}$$

Substitute above: $\boxed{\begin{array}{l} w = x+3 \\ dw = dx \end{array}}$

3. Compute $\displaystyle\int_2^\infty \frac{x}{e^{3x}}\ dx = \lim_{t\to\infty}\int_2^t \frac{x}{e^{3x}}\ dx = \lim_{t\to\infty}\int_2^t xe^{-3x}\ dx$

$$\overset{\text{IBP}}{=} \lim_{t\to\infty} -\frac{x}{3e^{3x}}\bigg|_2^t - \int_2^t -\frac{1}{3}e^{-3x}\ dx = \lim_{t\to\infty} -\frac{x}{3e^{3x}}\bigg|_2^t + \int_2^t \frac{1}{3}e^{-3x}\ dx$$

$$= \lim_{t\to\infty} -\frac{x}{3e^{3x}}\bigg|_2^t - \frac{1}{9e^{3x}}\bigg|_2^t$$

$$\lim_{t\to\infty} -\frac{t}{3e^{3t}} \overset{\frac{\infty}{\infty}}{} - \left(-\frac{2}{3e^6}\right) - \left(\frac{1}{9e^{3t}} - \frac{1}{9e^6}\right)$$

$$\overset{\text{L'H}}{=} \lim_{t\to\infty} -\frac{1}{9e^{3t}} + \frac{2}{3e^6} - 0 + \frac{1}{9e^6}$$

$$= 0 + \frac{2}{3e^6} + \frac{1}{9e^6} = \frac{6}{9e^6} + \frac{1}{9e^6} = \boxed{\frac{7}{9e^6}}$$

IBP: $\boxed{\begin{array}{ll} u = x & dv = e^{-3x}dx \\ du = dx & v = -\frac{1}{3}e^{-3x} \end{array}}$

4. Compute $\displaystyle\int_3^\infty \frac{7}{x^2+3x-10}\ dx = \lim_{t\to\infty}\int_3^t \frac{7}{x^2+3x-10}\ dx$

$$= \lim_{t\to\infty}\int_3^t \frac{7}{(x-2)(x+5)}\ dx \overset{\text{PFD}}{=} \lim_{t\to\infty}\int_3^t \frac{1}{x-2} - \frac{1}{x+5}\ dx$$

$$= \lim_{t\to\infty} \ln|x-2| - \ln|x+5|\bigg|_3^t = \lim_{t\to\infty} \ln|t-2| - \ln|t+5|\ ^{(\infty-\infty)} - (\ln(1) - \ln 8)$$

$$= \ln\left|\lim_{t\to\infty}\frac{t-2}{t+5}\right| + \ln 8 \overset{\frac{\infty}{\infty}}{}\overset{\text{L'H}}{=} \ln\left|\lim_{t\to\infty}\frac{1}{1}\right| + \ln 8 = 0 + \ln 8 = \boxed{\ln 8}$$

Partial Fractions Decomposition:

$$\frac{7}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

Clearing the denominator yields:

$7 = A(x+5) + B(x-2)$
$7 = (A+B)x + 5A - 2B$
so that $A + B = 0,$ and $5A - 2B = 7$
Solve for $A = 1,$ and $B = -1$

5. Compute $\displaystyle\int_8^\infty \frac{1}{x^2-10x+28}\,dx = \lim_{t\to\infty}\int_8^t \frac{1}{(x-5)^2+3}\,dx$ complete the square

Substitute

$u = x-5$	$x=8 \Rightarrow u=3$
$du = dx$	$x=t \Rightarrow u=t-5$

$$= \lim_{t\to\infty}\int_3^{t-5}\frac{1}{u^2+3}\,du = \lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_3^{t-5}$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-5}{\sqrt{3}}\right)-\arctan\left(\frac{3}{\sqrt{3}}\right)\right)$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-5}{\sqrt{3}}\right)-\arctan\left(\sqrt{3}\right)\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{3\pi}{6}-\frac{2\pi}{6}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \boxed{\frac{\pi}{6\sqrt{3}}}$$

using the formula $\displaystyle\int\frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$