Math 121 | Self-Assessment Quiz #3 | September 21, 2022

Answer Key

- Please see the course webpage for the answer key.

1. Compute $\lim_{x\to\infty}\left(\frac{6}{x}\right)^{\frac{1}{2+\ln x}}$ $(0^0) = \lim_{x\to\infty} e^{\ln\left(\left(\frac{6}{x}\right)^{\frac{1}{2+\ln x}}\right)} = e^{\lim_{x\to\infty}\ln\left(\left(\frac{6}{x}\right)^{\frac{1}{2+\ln x}}\right)}$

$$= e^{\lim_{x\to\infty}\frac{\ln\left(\frac{6}{x}\right)}{2+\ln x}} \left(\frac{-\infty}{\infty}\right) \stackrel{\text{L'H}}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{\left(\frac{6}{x}\right)}\left(-\frac{6}{x^2}\right)}{\frac{1}{x}}} = e^{\lim_{x\to\infty}\frac{\left(\frac{x}{6}\right)\left(-\frac{6}{x^2}\right)}{\frac{1}{x}}}$$

$$= e^{\lim_{x\to\infty}\frac{-\frac{1}{x}}{\frac{1}{x}}} = \boxed{e^{-1}}$$

2. Compute $\lim_{x\to 0}\frac{\cos(4x)-1-\arctan(4x)+4x}{\ln(1-x)+\arcsin x}\left(\frac{0}{0}\right) \stackrel{\text{L'H}}{=} \lim_{x\to 0}\frac{-4\sin(4x)-\frac{4}{1+16x^2}+4}{-\frac{1}{1-x}+\frac{1}{\sqrt{1-x^2}}}\left(\frac{0}{0}\right)$

$$\stackrel{\text{L'H}}{=} \lim_{x\to 0}\frac{-16\cos(4x)+\frac{4(32x)}{(1+16x^2)^2}}{-\frac{1}{(1-x)^2}-\frac{1}{2(1-x)^{\frac{3}{2}}}(-2x)} = \frac{-16+0}{-1+0} = \boxed{16}$$

3. Compute $\lim_{x\to\infty}\left(\sqrt{\frac{x^3-1}{x^3}}\right)^{x^3}$ $(1^\infty) = \lim_{x\to\infty} e^{\ln\left(\left(\sqrt{\frac{x^3-1}{x^3}}\right)^{x^3}\right)}$

$$= e^{\lim_{x\to\infty}\ln\left(\left(\sqrt{\frac{x^3-1}{x^3}}\right)^{x^3}\right)} = e^{\lim_{x\to\infty}x^3\ln\left(\sqrt{\frac{x^3-1}{x^3}}\right)^{\infty\cdot 0}}$$

$$= e^{\lim_{x\to\infty}\frac{\ln\left(\sqrt{1-\frac{1}{x^3}}\right)}{\frac{1}{x^3}}\frac{0}{0}} \stackrel{\text{L'H}}{=} e^{\lim_{x\to\infty}\frac{\left(\frac{1}{1-\frac{1}{x^3}}\right)\left(\frac{1}{2\sqrt{1-\frac{1}{x^3}}}\right)\left(\frac{3}{x^4}\right)}{-\frac{3}{x^4}}}$$

$$= e^{\lim_{x\to\infty} -\left(\frac{1}{2\left(1-\frac{1}{x^3}\right)}\right)} = e^{-\frac{1}{2}} = \boxed{\frac{1}{\sqrt{e}}}$$

OR you can convert the square root to a $\frac{1}{2}$ power.

$$\lim_{x\to\infty}\left(\sqrt{\frac{x^3-1}{x^3}}\right)^{x^3} (1^\infty) = \lim_{x\to\infty} e^{\ln\left(\left(\frac{x^3-1}{x^3}\right)^{\frac{x^3}{2}}\right)}$$

$$= e^{\lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x^3}\right)}{\frac{2}{x^3}}^{\frac{0}{0}}} \overset{\text{L'H}}{=} e^{\lim_{x\to\infty}\frac{\left(\frac{1}{1-\frac{1}{x^3}}\right)\left(\frac{3}{x^4}\right)}{-\frac{6}{x^4}}} = e^{-\frac{1}{2}} = \boxed{\frac{1}{\sqrt{e}}}$$

4. Compute $\int x \arcsin x \; dx = \frac{x^2}{2}\arcsin\; x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\; dx$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\; d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cdot\cos\theta\; d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta\; d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2}d\theta$$

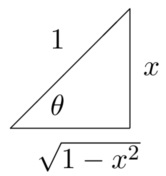
$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1-\cos(2\theta)d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}2\sin\theta\cos\theta + C$$

$$= \boxed{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$$

$u = \arcsin x$	$dv = xdx$
$du = \frac{1}{\sqrt{1-x^2}}dx$	$v = \frac{x^2}{2}$

Trig. Substitute $\boxed{\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta d\theta \end{aligned}}$



5. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{(\cos^2 x+1)^{\frac{7}{2}}}\ dx = -\int_1^0 \frac{1}{(u^2+1)^{\frac{7}{2}}}\ du = -\int_{\frac{\pi}{4}}^0 \frac{1}{(\tan^2\theta+1)^{\frac{7}{2}}} \cdot \sec^2\theta\ d\theta$

$$= -\int_{\frac{\pi}{4}}^0 \frac{1}{(\sec^2\theta)^{\frac{7}{2}}} \cdot \sec^2\theta\ d\theta = -\int_{\frac{\pi}{4}}^0 \frac{1}{(\sqrt{\sec^2\theta})^7} \cdot \sec^2\theta\ d\theta = -\int_{\frac{\pi}{4}}^0 \frac{1}{(\sec\theta)^7} \cdot \sec^2\theta\ d\theta$$

$$= -\int_{\frac{\pi}{4}}^0 \frac{\sec^2\theta}{\sec^7\theta}\ d\theta = -\int_{\frac{\pi}{4}}^0 \frac{1}{\sec^5\theta}\ d\theta = -\int_{\frac{\pi}{4}}^0 \cos^5\theta\ d\theta$$

$$= -\int_{\frac{\pi}{4}}^0 \cos^4\theta\cos\theta\ d\theta = -\int_{\frac{\pi}{4}}^0 (\cos^2\theta)^2\cos\theta\ d\theta = -\int_{\frac{\pi}{4}}^0 (1-\sin^2\theta)^2\cos\theta\ d\theta$$

$$= -\int_{\frac{1}{\sqrt{2}}}^0 (1-w^2)^2\ dw = -\int_{\frac{1}{\sqrt{2}}}^0 1-2w^2+w^4\ dw$$

$$= -\left(w - \frac{2w^3}{3} + \frac{w^5}{5}\right)\Bigg|_{\frac{1}{\sqrt{2}}}^0$$

$$= -\left(0-0+0-\left(\frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5\right)\right)$$

$$= \frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{2\sqrt{2}}\right) + \frac{1}{5}\left(\frac{1}{4\sqrt{2}}\right) = \frac{1}{\sqrt{2}} - \left(\frac{1}{3\sqrt{2}}\right) + \left(\frac{1}{20\sqrt{2}}\right)$$

$$= \frac{60}{60\sqrt{2}} - \left(\frac{20}{60\sqrt{2}}\right) + \left(\frac{3}{60\sqrt{2}}\right) = \boxed{\left(\frac{43}{60\sqrt{2}}\right)}$$

note that the minus sign can be removed in the second line by switching the limits of integration

$$= -\int_1^0 \frac{1}{(u^2+1)^{\frac{7}{2}}}\ du = \int_0^1 \frac{1}{(u^2+1)^{\frac{7}{2}}}\ du$$

OR finish this way with marking the original limits instead of changing them

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{(\cos^2 x+1)^{\frac{7}{2}}}\ dx = -\int_{x=0}^{x=\frac{\pi}{2}} \frac{1}{(u^2+1)^{\frac{7}{2}}}\ du \quad \ldots \quad = -\left(w - \frac{2w^3}{3} + \frac{w^5}{5}\right)\Bigg|_{x=0}^{x=\frac{\pi}{2}}$$

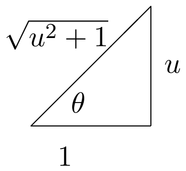
$$= -\left(\sin\theta - \frac{2\sin^3\theta}{3} + \frac{\sin^5\theta}{5}\right)\Bigg|_{x=0}^{x=\frac{\pi}{2}}$$

$$= -\left(\frac{u}{\sqrt{u^2+1}} - \frac{2u^3}{3(u^2+1)^{\frac{3}{2}}} + \frac{u^5}{5(u^2+1)^{\frac{5}{2}}}\right)\Bigg|_{x=0}^{x=\frac{\pi}{2}}$$

$$= -\left(\frac{\cos x}{\sqrt{\cos^2 x + 1}} - \frac{2\cos^3 x}{3(\cos^2 x+1)^{\frac{3}{2}}} + \frac{\cos^5 x}{5(\cos^2 x+1)^{\frac{5}{2}}}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= -\left(0 - 0 + 0 - \left(\frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5\right)\right) = \ldots \text{(as above)} \;= \boxed{\frac{43}{60\sqrt{2}}}$$

Standard u substitution to simplify at the start: $\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x dx \\ -du = \sin x dx \end{array}}$



Trig. Substitute $\boxed{\begin{array}{l} u = \tan\theta \\ du = \sec^2\theta d\theta \end{array}}$

Standard w substitution for *odd* trig. integral $\int \cos^5\theta\ d\theta$ technique:

$\boxed{\begin{array}{l} w = \sin\theta \\ dw = \cos\theta\ d\theta \end{array}}$

6. Compute $\int \left(\ln(x^2+1)\right)^2 \left(1 - \frac{1}{x^2}\right)\ dx$

$$\overset{\text{IBP}}{=} \left(x + \frac{1}{x}\right)\left(\ln(x^2+1)\right)^2 - 4\int \ln(x^2+1)\left(\frac{x^2+1}{x^2+1}\right)\ dx$$

$$= \left(x + \frac{1}{x}\right)\left(\ln(x^2+1)\right)^2 - 4\int \ln(x^2+1)\ dx$$

$$\overset{\text{IBP}}{=} \left(x + \frac{1}{x}\right)\left(\ln(x^2+1)\right)^2 - 4\left(x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\ dx\right)$$

$$= \left(x + \frac{1}{x}\right)\left(\ln(x^2+1)\right)^2 - 4x\ln(x^2+1) + 8\int \frac{x^2+1-1}{x^2+1}\ dx$$

$$= \left(x + \frac{1}{x}\right)\left(\ln(x^2+1)\right)^2 - 4x\ln(x^2+1) + 8\int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\ dx$$

$$= \left(x + \frac{1}{x}\right)\left(\ln(x^2+1)\right)^2 - 4x\ln(x^2+1) + 8\int 1 - \frac{1}{x^2+1}\ dx$$

$$= \boxed{\left(x + \frac{1}{x}\right)\left(\ln(x^2+1)\right)^2 - 4x\ln(x^2+1) + 8\left(x - \arctan x\right) + C}$$

Double IBP:

$u = \left(\ln(x^2+1)\right)^2$	$dv = 1 - \frac{1}{x^2}dx$
$du = 2\ln(x^2+1)\left(\frac{1}{x^2+1}\right)(2x)dx$	$v = x + \frac{1}{x}$

$u = \ln(x^2+1)$	$dv = 1dx$
$du = \frac{2x}{x^2+1}dx$	$v = x$