Extra Examples of Limits

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Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

1. $$\lim_{x\to 0} \frac{\arcsin x + \cos(3x) - e^x}{\arctan(3x) + x^2 - \sin(3x)} \overset{\left(\frac{0}{0}\right)\text{L'H}}{=} \lim_{x\to 0} \frac{\dfrac{1}{\sqrt{1-x^2}} - 3\sin(3x) - e^x}{\dfrac{3}{1+9x^2} + 2x - 3\cos(3x)} \left(\tfrac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\dfrac{1}{2(1-x^2)^{\frac{3}{2}}}(-2x) - 9\cos(3x) - e^x}{-\dfrac{3}{(1+9x^2)^2}(18x) + 2 + 9\sin(3x)}$$

$$= \lim_{x\to 0} \frac{\dfrac{x}{(1-x^2)^{\frac{3}{2}}} - 9\cos(3x) - e^x}{-\dfrac{54x}{(1+9x^2)^2} + 2 + 9\sin(3x)} = \frac{-9-1}{2} = \frac{-10}{2} = \boxed{-5}$$

2. $$\lim_{x\to 0} \frac{\ln(1-x) + x}{\cos(4x) - \arctan(3x) - e^{-3x}} \left(\tfrac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\dfrac{1}{1-x} + 1}{-4\sin(4x) - \dfrac{3}{1+9x^2} + 3e^{-3x}} \left(\tfrac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\dfrac{1}{(1-x)^2}}{-16\cos(4x) + \dfrac{3(18x)}{(1+9x^2)^2} - 9e^{-3x}} = \frac{-1}{-16-9} = \frac{-1}{-25} = \boxed{\frac{1}{25}}$$

3. $$\lim_{x\to 0^+} (1 - 3\sin x)^{\frac{1}{x}} \overset{1^\infty}{=} e^{\lim\limits_{x\to 0^+} \ln\left[(1-3\sin x)^{\frac{1}{x}}\right]}$$

$$= e^{\lim\limits_{x\to 0^+} \frac{\ln(1-3\sin x)}{x}} \left(\tfrac{0}{0}\right) \overset{\text{L'H}}{=} e^{\lim\limits_{x\to 0^+} \frac{\frac{1}{1-3\sin x}(-3\cos x)}{1}} = \boxed{e^{-3}}$$

4. $$\lim_{x\to\infty} (\ln x)^{\frac{3}{x}} \overset{\infty^0}{=} \lim_{x\to\infty} e^{\ln\left((\ln x)^{\frac{3}{x}}\right)} = e^{\lim\limits_{x\to\infty} \ln\left((\ln x)^{\frac{3}{x}}\right)} = e^{\lim\limits_{x\to\infty} \left(\frac{3}{x}\right)\ln(\ln x)}$$

$$e^{\lim\limits_{x\to\infty} \left(\frac{3\ln(\ln x)}{x}\right)} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=}} e^{\lim\limits_{x\to\infty} \left(\frac{\left(\frac{3}{\ln x}\right)\left(\frac{1}{x}\right)}{1}\right)} = e^{\lim\limits_{x\to\infty} \left(\frac{3}{x\ln x}\right)} = e^0 = \boxed{1}$$

$$5.\ \lim_{x\to\infty}\left(1-\arcsin\left(\frac{5}{x^4}\right)\right)^{3x^4} \overset{1^\infty}{=} e^{\lim\limits_{x\to\infty}\ln\left[\left(1-\arcsin\left(\frac{5}{x^4}\right)\right)^{3x^4}\right]}$$

$$= e^{\lim\limits_{x\to\infty} 3x^4\ln\left(1-\arcsin\left(\frac{5}{x^4}\right)\right)} \overset{\infty\cdot 0}{=} e^{\lim\limits_{x\to\infty}\frac{3\ln\left(1-\arcsin\left(\frac{5}{x^4}\right)\right)}{\frac{1}{x^4}}^{\left(\frac{0}{0}\right)}}$$

$$\overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\frac{\left(\frac{3}{1-\arcsin\left(\frac{5}{x^4}\right)}\right)\left(-\frac{1}{\sqrt{1-\left(\frac{5}{x^4}\right)^2}}\right)\left(-\frac{20}{x^5}\right)}{-\frac{4}{x^5}}}$$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{3}{1-\arcsin\left(\frac{5}{x^4}\right)}\right)\left(-\frac{1}{\sqrt{1-\left(\frac{5}{x^4}\right)^2}}\right)(5)} = \boxed{e^{-15}}$$

$$6.\ \lim_{x\to\infty}\left(1-\arcsin\left(\frac{6}{x}\right)\right)^{x} \overset{1^\infty}{=} \lim_{x\to\infty} e^{\ln\left(\left(1-\arcsin\left(\frac{6}{x}\right)\right)^{x}\right)}$$

$$= e^{\lim\limits_{x\to\infty}\ln\left(\left(1-\arcsin\left(\frac{6}{x}\right)\right)^{x}\right)} = e^{\lim\limits_{x\to\infty} x\ln\left(1-\arcsin\left(\frac{6}{x}\right)\right)^{\infty\cdot 0}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{\ln\left(1-\arcsin\left(\frac{6}{x}\right)\right)}{\frac{1}{x}}^{\left(\frac{0}{0}\right)}} \overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\frac{\left(\frac{1}{1-\arcsin\left(\frac{6}{x}\right)}\right)\left(-\frac{1}{\sqrt{1-\left(\frac{6}{x}\right)^2}}\right)\left(-\frac{6}{x^2}\right)}{-\frac{1}{x^2}}}$$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{1}{1-\arcsin\left(\frac{6}{x}\right)}\right)\left(-\frac{1}{\sqrt{1-\left(\frac{6}{x}\right)^2}}\right)(6)} = \boxed{e^{-6}}$$